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Determinant and Adjoint of Vector Valued Matrices

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Abstract. The products of odd-dimensional vectors are discussed in [40]. Based on these definitions, the product of vector valued matrix is investigated. In this paper, we studied the determinant and adjoint of vector valued matrices. The definitions are given and illustrated by examples.

Keywords: Vector valued matrix, determinant of vector valued matrices, adjoint of vector valued matrices

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1. Introduction

A matrix is a very important mathematical tool. It has various application in the field of Mathematics, Physics etc. This paper is the continuation of the previous published paper [40]. In this present article, the determinant and adjoint of vector valued matrices are defined and investigated.

Specially for the case of matrix multiplication, we have used generalised concept of odd multi-dimensional vector cross product by generalizing Eckmann [2] axioms which was given by Tian et al. [12]. Here, vector cross product in n-dimensional vector space is defined. Multi-dimensional vector product is defined by Silagadze [14]. For other works on vector product see [1,2,3,4,7,9,10,13] and multi-dimensional vector product are used in [5,6,8].

Several other types of matrices are available on fuzzy setup. There are some limitations in dealing with uncertainties by fuzzy set. Pal et al. defined intuitionistic fuzzy determinant in 2001 [29] and intuitionistic fuzzy matrices (IFMs) in 2002 [30]. Bhowmik and Pal [19] introduced some results on IFMs, intuitionistic circulant fuzzy matrix and generalized intuitionistic fuzzy matrix [19-25]. Shyamal and Pal [36-38] defined the distances between IFMs and hence defined a metric on IFMs. They also cited few applications of IFMs. In [28], the similarity relations, invertibility conditions and eigenvalues of IFMs are studied. Idempotent, regularity, permutation matrix and spectral radius of IFMs are also discussed. The parameterizations tool of IFM enhances the flexibility of its applications. For other works on IFMs see [16-18,27,33,34,37,38]. The concept of interval-valued fuzzy matrices (IVFMs) as a generalization of fuzzy matrix was introduced and developed in 2006 by Shaymal and Pal [39] by extending the max-min operation in fuzzy algebra. For more works on IVFMs see [32]. Combining IFMs and

IVFMs, a new fuzzy matrix called interval-valued intuitionistic fuzzy matrices (IVIFMs) is defined [26]. For other works on IVIFMs, see [23,25]. For recent works on uncertain matrix theory see [41-44].

1.1. Definition of odd multi-dimensional vector valued matrix

A rectangular array of mn elements A_{ij} into m rows and n columns, where the elements A_{ij}'s are the vectors i.e. of the form $(a_1^{ij}, a_2^{ij}, \ldots, a_k^{ij})$ where $a_m^{ij} \in F$(scalar field), belong to a vector space $V^K$of k dimension is called an odd multi-dimensional vector valued matrix for $k \geq 2$.

A $m \times n$ order k-dimensional vector valued matrix is exhibited in the form

$$\begin{bmatrix} (a_1^{11}, a_2^{11}, \ldots a_k^{11}) & (a_1^{12}, a_2^{12}, \ldots a_k^{12}) & \cdots & (a_1^{1n}, a_2^{1n}, \ldots a_k^{1n}) \\ (a_1^{21}, a_2^{21}, \ldots a_k^{21}) & (a_1^{22}, a_2^{22}, \ldots a_k^{22}) & \cdots & (a_1^{2n}, a_2^{2n}, \ldots a_k^{2n}) \\ \cdots & \cdots & & \cdots \\ (a_1^{n1}, a_2^{n1}, \ldots a_k^{n1}) & (a_1^{n2}, a_2^{n2}, \ldots a_k^{n2}) & \cdots & (a_1^{nn}, a_2^{nn}, \ldots a_k^{nn}) \end{bmatrix}$$

1.3. Various type of vector valued matrices

Row and column vector valued matrix

In a $m \times n$ VVM (vector valued matrix (VVM) if $m = 1$, then the VVM is called r*ow VVM*.

e.g. $\begin{bmatrix} (1,2,3) & (0,0,1) & (1,0,1) \end{bmatrix}$ etc.

When $n = 1$, then the VVM is called column VVM.

e.g. $\begin{bmatrix} (1,4,5) \\ (2,0,1) \\ (0,0,4) \end{bmatrix}$ etc.

Null vector valued matrix

If each element of a VVM be zero vector then the VVM is called *Null VVM*. A $m \times n$ order k-dimensional Null VVM is denoted as $O_{m,n}$.

Square vector valued matrix

A VVM is said to be a *square VVM* if the number of rows of it is equal to the number of column of it.

Diagonal vector valued matrix

A Square VVM is called *Diagonal matrix*,if all of its non-diagonal elements are zero vector.

Identity or unit vector valued matrix

A Square VVM is said to be *Identity* or *Unit VVM*, if all diagonal elements of it are equal to unit vector and non-diagonal elements are all zero vectors. A $n \times n$ order k-dimensional VVM is denoted as I_n^k.

Upper and lower triangular vector valued matrix

A Square VVM is said to be *Upper Triangular VVM* if all elements of it below the leading diagonal are zero vectors and it is said to be *Lower Triangular VVM* if all elements of it above the leading diagonal are zero vectors.

1.3. Various type of algebraic operations on vector valued matrices

We consider Vector Valued Matrices of same dimension.

Addition

Two vector valued matrices A and B are said be Conformal for Addition if they have same order.

If $A = (a_1^{ij}, a_k^{ij}, \dots a_k^{ij})_{m,n}$ and $B = (b_1^{ij}, b_k^{ij}, \dots b_k^{ij})_{m,n}$ be two k dimensional vector valued matrices of order $\times n$. Then their sum is a k dimensional vector valued matrix C of order $m \times n$ and it is defined as

$$C = (c_1^{ij}, c_2^{ij}, \dots, c_k^{ij})_{m,n} = (a_1^{ij} + b_1^{ij}, a_2^{ij} + b_2^{ij}, \dots, a_k^{ij} + b_k^{ij})_{m,n}$$

If A and B two vector valued matrices of different order and different dimensions then Addition is not defined.

Subtraction

Two vector valued matrices A and B are said be Conformal for Subtraction if they have same order.

If $A = (a_1^{ij}, a_k^{ij}, \dots a_k^{ij})_{m,n}$ and $B = (b_1^{ij}, b_k^{ij}, \dots b_k^{ij})_{m,n}$ be two k(odd) dimensional vector valued matrices of order $\times n$. Then their difference is an k(odd) dimensional vector valued matrix C of order $m \times n$ and it is defined as

$$C = (c_1^{ij}, c_2^{ij}, \dots, c_k^{ij})_{m,n} = (a_1^{ij} - b_1^{ij}, a_2^{ij} - b_2^{ij}, \dots, a_k^{ij} - b_k^{ij})_{m,n}$$

Scalar multiplication

The product of a $m \times n$ order k-dimensional VVM , $A = (a_1^{ij}, a_k^{ij}, \dots a_k^{ij})_{m,n}$by a scalar c where $c \in F$, the field of scalars, is a $m \times n$ order k(odd)-dimensional VVM, $B = (b_1^{ij}, b_k^{ij}, \dots b_k^{ij})_{m,n}$ defined by

$\left(b_1^{ij}, b_k^{ij}, \dots b_k^{ij}\right) = \left(c \cdot a_1^{ij}, c \cdot a_k^{ij}, \dots c \cdot a_k^{ij}\right), i = 1,2, \cdots, m; j = 1,2, \cdots, n;$ and it can be written as $B = cA$.

Let A be a $m \times n$ order odd-dimensional VVM and c, d are scalars.Then the following results are obvious.

a) $c(dA) = (cd)A$,
b) $0A = O_{m,n}$; 0 being the zero element of$F$,
c) $cO_{m,n} = O_{m,n}$,
d) $1A = A$, 1 being the identity element of F.

Multiplication

Here the matrices are of vector valued. Hence we define two type of multiplication for such matrices. These are scalar multiplication (dot product) and vector multiplication (cross product).

Dot product between two VVM

Two VVM, A and B are said to be conformal for the dot product if they have the same dimensions and the number of columns of A is equal to the number of rows of B. If $A = (a_1^{ij}, a_k^{ij}, \dots a_k^{ij})_{m,n}$and $B = (b_1^{ij}, b_k^{ij}, \dots b_k^{ij})_{n,p}$ then the dot product between A and B (denoted as $A \cdot B$) is a scalar matrix C of order $m \times p$ defined as,

$$A \cdot B = C = (c_1^{ij}, c_2^{ij}, \dots, c_k^{ij})_{m,p} \text{ where } \left(c_1^{ij}, c_2^{ij}, \dots, c_k^{ij}\right) = \sum_{t=1}^{n} (a_1^{ij}, a_2^{ij}, \dots, a_k^{ij}) \times (b_1^{ij}, b_2^{ij}, \dots, b_k^{ij}) = \left(a_1^{it} \cdot b_1^{tj}, a_2^{it} \cdot b_2^{tj}, \dots, a_k^{it} . b_k^{tj}\right),\ i = 1,2, \cdots, m;\ j = 1,2, \cdots, p.$$

2. Transpose of a vector valued matrix

Let A be a $m \times n$ order k(odd)-dimensional VVM . Then the $n \times m$ VVM obtain by interchanging rows and columns of A is said to be the transpose of A and it is denoted by $A^t (or\ A^T)$.

Thus if $A = (a_1^{ij}, a_k^{ij}, \dots a_k^{ij})_{m,n}$ then $A^t = B = (b_1^{ij}, b_k^{ij}, \dots b_k^{ij})_{n,m}$ where $\left(b_1^{ij}, b_k^{ij}, \dots b_k^{ij}\right) = (a_1^{ji}, a_k^{ji}, \dots a_k^{ji})$, $i = 1,2, \dots, n; j = 1,2, \dots, m$.

Example 2.1. Consider a 2×2 order 3-dimensional VVM, $A = \begin{bmatrix} (1,0,0) & (0,1,0) \\ (1,2,1)) & (0,0,1) \end{bmatrix}$ then

$$A^t = \begin{bmatrix} (1,0,0) & (1,2,1) \\ (0,1,0) & (0,0,1) \end{bmatrix}$$

Theorem 3.1. $(A^t)^t = A$.

Theorem 3.2. If A and B two odd-dimensional VVMs such that $A + B$ is defined then

$$(A + B)^t = A^t + B^t.$$

Theorem 3.3. If c is a scalar, $(cA)^t = cA^t$.

Theorem 3.4. If A and B two odd-dimensional VVMs such that AB is defined then $(AB)^t = B^t A^t$.

The proofs of the above theorems are obvious.

Symmetric and skew symmetric vector valued matrices

A square odd-dimensional VVM A is said to be s*ymmetric* VVM if $A = A^t$. Therefore $A = (a_1^{ij}, a_2^{ij}, \dots a_k^{ij})_{m,n}$ is *Symmetric* if $\left(a_1^{ij}, a_2^{ij}, \dots a_k^{ij}\right) = (a_1^{ji}, a_2^{ji}, \dots a_k^{ji})$.

Example 2.2. $\begin{bmatrix} (1,0,0) & (1,2,3) & (2,9,5) \\ (1,2,3) & (0,1,0) & (1,5,1) \\ (2,9,5) & (1,5,1) & (0,0,1) \end{bmatrix}$.

A square odd-dimensional VVM A is said to be *skew-symmetric* VVM if $A = -A^t$. Therefore $A = (a_1^{ij}, a_2^{ij}, \ldots a_k^{ij})_{m,n}$ is *skew-symmetric* if $\left(a_1^{ij}, a_2^{ij}, \ldots a_k^{ij}\right) = -(a_1^{ji}, a_2^{ji}, \ldots a_k^{ji})$.

Example 2.3. $\begin{bmatrix} (1,0,0) & (1,2,1) & (2,9,5) \\ (-1,-2,-1) & (0,1,0) & (-1,-4,-1) \\ (-2,-9,-5) & (1,4,1) & (0,0,1) \end{bmatrix}$.

Some important *results* hold for such type of VVMs given as

a) If A and B two symmetric VVM of same order then $A + B$ is symmetric.
b) If A and B two symmetric VVM of same order then AB is symmetric if and only if $AB = BA$.
c) If A be a $m \times n$ VVM, then the VVMs AA^t and A^tA are both symmetric.
d) If A real square VVM can be uniquely expressed as the sum of symmetric VVM and skew symmetric VVM.

3. Determinant

Here we generalized the idea of determinant in case of vector valued matrices which is defined as follows:

Definition 3.1. A determinant function $f: S \to V$ is a vector valued function on the set S of all$n \times n$ ordervector valued matrices of k(odd)-dimension over the vector space V such that if $A = (a_1^{ij}, a_2^{ij}, \ldots, a_k^{ij}) \in S$, then $f(A)$, or $\det A$ is a vectorbelonging to V and is defined by

$$\det A = \sum_{\varphi} sgn\, \varphi \left(a_1^{1\varphi(1)}, a_2^{1\varphi(1)}, \ldots a_k^{1\varphi(1)}\right) \times \left(a_1^{2\varphi(2)}, a_2^{2\varphi(2)}, \ldots a_k^{2\varphi(2)}\right) \times \cdots \times \left(a_1^{n\varphi(n)}, a_2^{n\varphi(n)}, \ldots a_k^{n\varphi(n)}\right),$$

whereφ is a permutation on $\{1,2,\ldots,n\}$ and $sgn\, \varphi = 1\; or - 1$ according as the permutation

$$\varphi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \varphi(1) & \varphi(1) & \cdots & \varphi(1) \end{pmatrix} \text{ is even or odd.}$$

$\det A$ is said to be a determinant of order n and is denoted by the symbol

$$\begin{vmatrix} \left(a_1^{11}, a_2^{11}, \ldots a_k^{11}\right) & \left(a_1^{12}, a_2^{12}, \ldots a_k^{12}\right) & \cdots & \left(a_1^{1n}, a_2^{1n}, \ldots a_k^{1n}\right) \\ \left(a_1^{21}, a_2^{21}, \ldots a_k^{21}\right) & \left(a_1^{22}, a_2^{22}, \ldots a_k^{22}\right) & \cdots & \left(a_1^{2n}, a_2^{2n}, \ldots a_k^{2n}\right) \\ \cdots & \cdots & & \cdots \\ \left(a_1^{n1}, a_2^{n1}, \ldots a_k^{n1}\right) & \left(a_1^{n2}, a_2^{n2}, \ldots a_k^{n2}\right) & \cdots & \left(a_1^{nn}, a_2^{nn}, \ldots a_k^{nn}\right) \end{vmatrix}$$

Or shortly by$\left|\left(a_1^{ij}, a_2^{ij}, \ldots a_k^{ij}\right)\right|_n$.

The summation is said to be the expansion of $\det A$. It contains $n!$ terms as there are $n!$ permutations on the set $\{1,2,\dots,n\}$.As there are $\frac{1}{2}n!$ even and $\frac{1}{2}n!$ odd permutations on the set$\{1,2,\dots,n\}$, contains $\frac{1}{2}n!$ positive terms and $\frac{1}{2}n!$ negative terms.

Each term is a cross product of n vector elements. In each term the first suffices (row suffices) of the elements appears in their natural order and the second suffices appears in a permutation of 1,2,…,n. So each term contains one element from each row and one element from each column of A.

We now illustrate determinant of vector valued matrices by following examples

Examples 3.1. Let us consider a 3×3 order 3-dimensional real vector valued matrix given by

$$A = \begin{bmatrix} (1,1,0) & (2,1,3) & (0,1,0) \\ (1,1,3) & (1,0,1) & (1,1,0) \\ (2,0,1) & (2,3,4) & (1,3,1) \end{bmatrix} \text{then}$$

$$\det \mathrm{A} = \begin{vmatrix} (1,1,0) & (2,1,3) & (0,1,0) \\ (1,1,3) & (1,0,1) & (1,1,0) \\ (2,0,1) & (2,3,4) & (1,3,1) \end{vmatrix}$$

$$\begin{aligned} &= (1,1,0) \times [(1,0,1) \times (1,3,1) - (1,1,0) \times (2,3,4)] \\ &+ (2,1,3) \times [(1,1,0) \times (2,0,1) - (1,1,3) \times (1,3,1)] \\ &+ (0,1,0) \times [(1,1,3) \times (2,3,4) - (1,0,1) \times (2,0,1)] \\ &= (1,1,0) \times [(-3,0,3) - (4,-4,1)] \\ &+ (2,1,3) \times [(1,-1,-2) - (-8,2,2)] + (0,1,0) \times [(-5,2,1) - (0,1,0)] \end{aligned}$$

$$\begin{aligned} &= (1,1,0) \times (-7,4,2) + (2,1,3) \times (9,-3,-4) + (0,1,0) \times (-5,1,1) \\ &= (2,-2,11) + (5,35,-15) + (1,0,5) = (8,33,1) \end{aligned}$$

Hence, $\det A = (8,33,1)$.

Example 3.2. Let us consider a 2×2 order 5-dimensional real vector valued matrix given by

$$B = \begin{bmatrix} (1,1,2,3,4) & (2,3,1,1,2) \\ (1,0,1,0,3)) & (4,1,0,0,1) \end{bmatrix} \text{ then}$$

$$\det B = \begin{vmatrix} (1,1,2,3,4) & (2,3,1,1,2) \\ (1,0,1,0,3)) & (4,1,0,0,1) \end{vmatrix}$$

$$= (1,1,2,3,4) \times (4,1,0,0,1) - (2,3,1,1,2) \times (1,0,1,0,3)$$

Now to compute the cross product we used (14) and get

$$\begin{aligned} &(1,1,2,3,4) \times (4,1,0,0,1) \\ &= \left\{ \left[\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \right.\right. \\ &\left.\left. + \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} \right] \right\} = (-1,11,15,13,-3) \end{aligned}$$

$(2,3,1,1,2) \times (1,0,1,0,3)$

$$= \left\{ \left[\begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} \right], \left[\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \right] \right\} = (1,2,-8,-1,-4)$$

Hence, $\det A = (-1,11,15,13,-3) - (1,2,-8,-1,-4) = (-2,9,23,14,1)$

Example 3.3. Let us consider a 2×2 order 7-dimensional real vector valued matrix given by

$$C = \begin{bmatrix} (1,2,1,0,0,1,2) & (2,3,4,1,2,1,0) \\ (1,0,0,1,2,3,4) & (2,1,0,1,0,2,0) \end{bmatrix}.$$

Now, $\det C = \begin{vmatrix} (1,2,1,0,0,1,2) & (2,3,4,1,2,1,0) \\ (1,0,0,1,2,3,4) & (2,1,0,1,0,2,0) \end{vmatrix}$

$= (1,2,1,0,0,1,2) \times (2,1,0,1,0,2,0) - (2,3,4,1,2,1,0) \times (1,0,0,1,2,3,4)$

Now elementary cross products in 7-dimension, are calculated with the help of 2nd algorithm of Table 5 as follows:

$(1,2,1,0,0,1,2) \times (2,1,0,1,0,2,0)$

$$= \left\{ \left[\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} \right], \left[\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \right], \left[\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \right] \right\}$$
$$= (2,-5,3,-5,0,3,1)$$

Similarly, $(2,3,4,1,2,1,0) \times (1,0,0,1,2,3,4)$

$$= \left\{ \left[\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} \right], \left[\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \right], \left[\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 4 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} \right], \left[\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} \right], \left[\begin{vmatrix} 2 & 4 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \right] \right\}$$
$$= (25,4,-10,-15,-9,-6,5)$$

Hence, $\det C = (2,-5,3,-5,0,3,1) - (25,4,-10,-15,-9,-6,5)$
$= (-23,-9,13,10,9,9,-4)$.

Generally, the results hold for usual determinant of matrices may or may not holds for the determinant of VVM, for example, the result $\det A = detA^t$ holds for the determinants of matrices but does not holds for odd dimensional vector valued matrices since vector cross multiplication is not commutative.

We can easily illustrate this by considering a 2×2 order 3-dimensional matrix

$$A = \begin{bmatrix} (1,0,1) & (1,2,3) \\ (2,1,0) & (4,2,1) \end{bmatrix}.$$

Then $detA = \begin{vmatrix} (1,0,1) & (1,2,3) \\ (2,1,0) & (4,2,1) \end{vmatrix} = (1,0.1) \times (4,2,1) - (1,2,3) \times (2,1,0)$
$= (-2,3,2) — (3,6,-3) = (1,-3,5).$

But $detA^t = \begin{vmatrix} (1,0,1) & (2,1,0) \\ (1,2,3) & (4,2,1) \end{vmatrix} = (1,0.1) \times (4,2,1) - (2,1,0) \times (1,2,3)$
$= (-2,3,2) — (3,-6,3) = (-5,9,-1).$

Hence, clearly $\det A \neq detA^t$.

4. Cofactor and minors

Let $A = (a_1^{ij}, a_2^{ij}, \dots, a_k^{ij})$ be a $n \times n$ order k-dimensional VVM.

Then $\left(a_1^{1\varphi(1)}, a_2^{1\varphi(1)}, \dots a_k^{1\varphi(1)}\right) \times \left(a_1^{2\varphi(2)}, a_2^{2\varphi(2)}, \dots a_k^{2\varphi(2)}\right) \times \cdots \times$
$\left(a_1^{n\varphi(n)}, a_2^{n\varphi(n)}, \dots a_k^{n\varphi(n)}\right)$, where

φ is a permutation on $\{1,2,\dots,n\}$ and $sgn\ \varphi = 1\ or - 1$ according as the permutation

$$\varphi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \varphi(1) & \varphi(1) & \cdots & \varphi(1) \end{pmatrix} \text{ is even or odd.}$$

There are $n!$ terms in the expression. Each term contains one and only one element from each row and one and only one component from each column.

Let us consider, in particular, the ith row. Each term of the expansion of $\det A$ contains one and only of $a_{1n}, a_{2n}, \dots, a_{in}$. Therefore the expansion of $detA$ can be exhibited as,

$$detA = \left(a_1^{i1}, a_2^{i1}, \dots, a_k^{i1}\right)(***) + \left(a_1^{i2}, a_2^{i2}, \dots, a_k^{i2}\right)(***) + \cdots + \left(a_1^{in}, a_2^{in}, \dots, a_k^{in}\right)(***).$$

The companion factors $(***)$ of $(a_1^{ij}, a_2^{ij}, \dots, a_k^{ij})$ is called the *cofactor* of $(a_1^{ij}, a_2^{ij}, \dots, a_k^{ij})$ in $\det A$ and is denoted by A_{ij}.

Thus

$$detA = \left(a_1^{i1}, a_2^{i1}, \dots, a_k^{i1}\right)A_{i1} + \left(a_1^{i2}, a_2^{i2}, \dots, a_k^{i2}\right)A_{i2} + \cdots + \left(a_1^{in}, a_2^{in}, \dots, a_k^{in}\right)A_{in}.$$

Again if one row and one column be deleted from an $n \times n$ order $k(odd)$-dimensional VVM , $A = (a_1^{ij}, a_2^{ij}, \dots, a_k^{ij})$, the determinant of the remaining $(n-1) \times (n-1)$ matrix is said to be a minor of order $(n-1)$ of A. The minor of order $(n-1)$ obtained by deleting ith row and jth column is denoted by M_{ij} and is said to be the minor of the element $(a_1^{ij}, a_2^{ij}, \dots, a_k^{ij})$ of $\det A$.

In the determinant

$$\begin{vmatrix} \left(a_1^{11}, a_2^{11}, \dots a_k^{11}\right) & \left(a_1^{12}, a_2^{12}, \dots a_k^{12}\right) & \cdots & \left(a_1^{1n}, a_2^{1n}, \dots a_k^{1n}\right) \\ \left(a_1^{21}, a_2^{21}, \dots a_k^{21}\right) & \left(a_1^{22}, a_2^{22}, \dots a_k^{22}\right) & \cdots & \left(a_1^{2n}, a_2^{2n}, \dots a_k^{2n}\right) \\ \cdots & \cdots & & \cdots \\ \left(a_1^{n1}, a_2^{n1}, \dots a_k^{n1}\right) & \left(a_1^{n2}, a_2^{n2}, \dots a_k^{n2}\right) & \cdots & \left(a_1^{nn}, a_2^{nn}, \dots a_k^{nn}\right) \end{vmatrix},$$

$$M_{11} = \begin{vmatrix} (a_1^{22}, a_2^{22}, \dots a_k^{22}) & (a_1^{23}, a_2^{23}, \dots a_k^{23}) \\ (a_1^{32}, a_2^{32}, \dots a_k^{32}) & (a_1^{33}, a_2^{33}, \dots a_k^{33}) \end{vmatrix},$$

$$M_{12} = \begin{vmatrix} (a_1^{21}, a_2^{21}, \dots a_k^{21}) & (a_1^{23}, a_2^{23}, \dots a_k^{23}) \\ (a_1^{31}, a_2^{31}, \dots a_k^{31}) & (a_1^{33}, a_2^{33}, \dots a_k^{33}) \end{vmatrix},$$

$$M_{13} = \begin{vmatrix} (a_1^{21}, a_2^{21}, \dots a_k^{21}) & (a_1^{22}, a_2^{22}, \dots a_k^{22}) \\ (a_1^{31}, a_2^{31}, \dots a_k^{31}) & (a_1^{32}, a_2^{32}, \dots a_k^{32}) \end{vmatrix}$$, etc are minors of the above determinant.

5. Adjoint of VVM

Let $A = (a_1^{ij}, a_2^{ij}, \dots, a_k^{ij})$ be a square $k(odd)$-dimensional VVM.. Let A_{ij} be the cofactor of $(a_1^{ij}, a_2^{ij}, \dots, a_k^{ij})$ in $\det A$. The transpose of the matrix (A_{ij}) is said to be the *adjoint (or adjugate)* of A and is denoted by $adj\ A$.

Example 5.1. Let us consider a 3-dimensional 3×3 order VVM defined as,

$$A = \begin{bmatrix} (1,0,1) & (1,2,3) & (2,1,0) \\ (2,1,3) & (1,0,0) & (4,0,1) \\ (3,0,6) & (6,3,0) & (0,1,0) \end{bmatrix}.$$

Then, $$adj\ A = \begin{vmatrix} +\begin{vmatrix} (1,0,1) & (4,0,1) \\ (6,3,0) & (0,1,0) \end{vmatrix} & -\begin{vmatrix} (1,2,3) & (2,1,0) \\ (6,3,0) & (0,1,0) \end{vmatrix} & +\begin{vmatrix} (1,2,3) & (2,1,0) \\ (1,0,0) & (4,0,1) \end{vmatrix} \\ -\begin{vmatrix} (2,1,3) & (4,0,1) \\ (3,0,6) & (0,1,0) \end{vmatrix} & +\begin{vmatrix} (1,0,1) & (2,1,0) \\ (3,0,6) & (0,1,0) \end{vmatrix} & -\begin{vmatrix} (1,0,1) & (2,1,0) \\ (2,1,3) & (4,0,1) \end{vmatrix} \\ +\begin{vmatrix} (2,1,3) & (1,0,0) \\ (3,0,6) & (6,3,0) \end{vmatrix} & -\begin{vmatrix} (1,0,1) & (1,2,3) \\ (3,0,6) & (6,3,0) \end{vmatrix} & +\begin{vmatrix} (1,0,1) & (1,2,3) \\ (2,1,3) & (1,0,0) \end{vmatrix} \end{vmatrix}.$$

$$Adj\ A = \begin{bmatrix} (3,-6,12) & (3,0,-1) & (2,11,-7) \\ (3,-21,2) & (-7,12,4) & (3,-9,0) \\ (-9,24,0) & (15,-3,-9) & (-3,-2,3) \end{bmatrix}.$$

Properties:

(a) $adj\ (A^t) = (adj\ A)^t$.

(b) If A be an $n \times n$ order $k(odd)$-dimensional VVM and c be a scalar. Then $adj\ (cA) = c^{n-1} adj\ A$.

Proof of those aforesaid properties are oblivious.

6. Conclusion

Based on the algorithms to find product of odd dimensional vectors, we defined determinant and adjoint of a square odd-dimensional vector valued matrix. Although, still we are unable to define the inverse of a vector valued matrix because the determinant of an odd-dimensional vector valued matrix is generalized into a vector valued function instead of scalar function so further investigation is required in this field. Also, its potential application in mathematics and physics also deserves to be further investigated.,

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