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Progress in Nonlinear Dynamics and Chaos

Solution of System of Linear Equations with Coefficients as Triangular Fuzzy Number

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Abstract. In the field of science and engineering technology, linear systems have contributed more applications. But, in fact, the linear systems occur in uncertain environment. In such situations, the parameter of the system can be represented by fuzzy nature with the help of fuzzy numbers. In this paper, we have to studied fuzzy linear systems with the aid of triangular fuzzy numbers. A new procedure namely matrix inversion method is proposed for solving fuzzy linear system (FLS) of equations. Finally, the method is illustrated by solving relevant numerical examples.

Keywords: triangular fuzzy numbers, system of linear equations, solution of fuzzy system of equations

AMS Mathematics Subject Classification (2010): 15B15

1. Introduction

Uncertainty can be classified into two types-probabilistic uncertainty and fuzzy uncertainty, though people were aware of fuzzy uncertainty before the mathematical formulation of fuzziness by Zadeh. Fuzziness can be represented in different ways. One of the most useful representation is membership function. Also, depending the nature or shape of membership function a fuzzy number can be classified in different ways, such as triangular fuzzy number (TFN), trapezoidal fuzzy number etc. Triangular fuzzy numbers (TFNs) are frequently used in applications. It is well known that the matrix formulation of a mathematical formula gives extra facility to handle/study the problem. Due to the presence of uncertainty in many mathematical formulations in different branches of science and technology, we introduce triangular fuzzy matrices (TFMs). To the best of our knowledge, no work is available on TFMs, though a lot of work on fuzzy matrices is available in literature. A brief review on fuzzy matrices is given below.

Fuzzy matrices were introduced for the first time by Thomason [4], who discussed the convergence of powers of fuzzy matrix. Xin studied the controllable fuzzy matrix. Ragab et al. [3] presented some properties of the min-max composition of fuzzy matrices. Kim et al. [2] presented some important results on determinant of a square fuzzy matrices.

Several other types of matrices are available on fuzzy setup. There are some limitations in dealing with uncertainties by fuzzy set. Pal et al. defined intuitionistic fuzzy determinant in 2001 [18] and intuitionistic fuzzy matrices (IFMs) in 2002 [19]. Bhowmik and Pal [8] introduced some results on IFMs, intuitionistic circulant fuzzy matrix and generalized intuitionistic fuzzy matrix [8-14]. Shyamal and Pal [25-27] defined the distances between IFMs and hence defined a metric on IFMs. They also cited few applications of IFMs. In [17], the similarity relations, invertibility conditions and eigenvalues of IFMs are studied. Idempotent, regularity, permutation matrix and spectral radius of IFMs are also discussed. The parameterizations tool of IFM enhances the flexibility of its applications. For other works on IFMs see [5-7,16,22,23,26,27]. The concept of interval-valued fuzzy matrices (IVFMs) as a generalization of fuzzy matrix was introduced and developed in 2006 by Shaymal and Pal [28] by extending the max-min operation in fuzzy algebra. For more works on IVFMs see [21]. Combining IFMs and IVFMs, a new fuzzy matrix called interval-valued intuitionistic fuzzy matrices (IVIFMs) is defined [15]. For other works on IVIFMs, see [12,14]. For recent works on uncertain matrix theory see [30-33].

In this article, a fuzzy system of linear equations is investigated where the coefficients are triangular fuzzy numbers. The matrix inverse method is discussed in this paper.

2. Preliminaries

In this section some basic related definitions are studied and recalled the representations of Fuzzy Numbers.

Definition 2.1. A fuzzy set is characterized by a membership function mapping the elements of a domain, space or universe of discourse to the unit interval [0,1].

A fuzzy set $\widetilde{A}$ in a universe of discourse X is defined as the following set of pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x): x \in X)\}$$

Here $\mu_{\tilde{A}}(x): X \to [0,1]$ is mapping called the degree of membership function of the fuzzy set $\tilde{A}$ and $\mu_{\tilde{A}}(x)$is called the membership value of $x \in X$ in the fuzzy set $\tilde{A}$. These membership grades are often represented by real ranging from [0,1].

Definition 2.2. Convex fuzzy set: A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)\} \subseteq X$ is called convex set in all A_α are convex set i.e. for every element $x_1 \in A_\alpha$ and $x_2 \in A_\alpha$ for every $\alpha \in [0,1]. \lambda x_1 + (1-\lambda)\, x_2 \in A_\alpha$ for all $\lambda \in [0,1]$. Otherwise the fuzzy set is called non-convex fuzzy set.

Definition 2.3. Fuzzy number: A fuzzy set $\tilde{A}$, defined on the set of real numbers R is said to be fuzzy number if its membership function has the following characteristics

1. $\tilde{A}$ is normal.
2. $\widetilde{A}$is convex set.
3. The support of $\tilde{A}$ is closed and bounded then $\tilde{A}$ is called fuzzy number.

3. Triangular fuzzy number

Sometimes it may happen that some data or numbers cannot be specified precisely or accurately due to the error of the measuring technique or instruments etc. Suppose the height of a person is recorded as $160\ cm$. However, it is impossible in practice to measure the height accurately; actually this height is about $160\ cm$; it may be a bit more or a bit less than $160\ cm$. Thus the height of that person can be written more precisely as the triangular fuzzy number$(160-\alpha, 160, 160+\beta)$, where α and β are the left and right spreads. In general, a TFN "a" can written as $(a-\alpha, a, a+\beta)$, where$\alpha$and β are the left and right spreads of a respectively. These type of numbers are alternately represented as $< a, \alpha, \beta >$. The mathematical definition of a TFN is given below.

Definition 3.1. A triangular fuzzy number denoted by$\widetilde{M} =< m, \alpha, \beta >$,has the membership function

$$\mu_{\widetilde{M}}(x) = \begin{cases} 0 & for\ x \leq m-\alpha \\ 1-\dfrac{m-x}{\alpha} & for\ m-\alpha < x < m \\ 1 & for\ x = m \\ 1-\dfrac{x-m}{\beta} & for\ m < x < m+\beta \\ 0 & for\ x \geq m+\beta \end{cases}$$

The point m, with membership grade of 1, is called the mean value and α, β are the left hand and right hand spreads of M respectively.

A TFN is said to be symmetric if both its spreads are equal, i.e., if $\alpha = \beta$ and it is sometimes denoted by $\widetilde{M} =< m, \alpha >$.

Due to the wide field of applications of TFNs, many authors have tried to define the basic arithmetic operations on TFNs. Here we introduce the definitions of arithmetic operations due to Dubois and Prade.

3.2. Arithmetic operations on TFNs

Let $\widetilde{M} =< m, \alpha, \beta >$ and $\widetilde{N} =< n, \gamma, \delta >$ be two TFNs.

(1) Addition: $\widetilde{M} + \widetilde{N} =< m+n, \alpha+\gamma, \beta+\delta >$

(2) Scalar multiplication: Let λ be a scalar, $\lambda\widetilde{M} = < \lambda m, \lambda\alpha, \lambda\beta >$, when $\lambda \geq 0$
$\lambda\widetilde{M} = < \lambda m, -\lambda\beta, -\lambda\alpha >$, when $\lambda \leq 0$. In particular, $-\widetilde{M} =< -m, \beta, \alpha >$.

(3) Subtraction: $\widetilde{M} - \widetilde{N} = < m-n, \alpha+\delta, \beta+\gamma >$.
For two TFNs $\widetilde{M}$ and $\widetilde{N}$, their addition, subtraction and scalar multiplication, i.e.,$\widetilde{M} + \widetilde{N}$, $\widetilde{M} - \widetilde{N}$and $\lambda\widetilde{M}$ are all TFNs.

(4) Multiplication: It can be shown that the shape of the membership function of $\widetilde{M}.\widetilde{N}$ is not necessarily a triangular, but, if the spreads of $\widetilde{M}$ and $\widetilde{N}$ are small compared to their mean values m and n then the shape of membership function is closed to a triangle. A good approximation is as follows:

(a) When $\widetilde{M} \geq 0$ and $\widetilde{N} \geq 0$ $(\widetilde{M} \geq 0, if\ m \geq 0)$

$$\widetilde{M}.\widetilde{N} =< m, \alpha, \beta >.< n, \gamma, \delta > \approx < mn, m\gamma + n\alpha, m\delta + n\beta >$$

(b) When $\widetilde{M} \leq 0$, $\widetilde{N} \geq 0$

$$\widetilde{M}.\widetilde{N} = < m, \alpha, \beta >.< n, \gamma, \delta > \approx < mn, n\alpha - m\delta, n\beta - m\gamma >.$$

(c) When $\widetilde{M} \leq 0$ and $\widetilde{N} \leq 0$

$$\widetilde{M}.\widetilde{N} = < m, \alpha, \beta >. < n, \gamma, \delta > \approx < mn, -n\beta - m\delta, -n\alpha - m\gamma >.$$

When spreads are not small compared with mean values, the following is a better approximation:

$< m, \alpha, \beta >. < n, \gamma, \delta > \approx < mn, m\gamma + n\alpha - \alpha\gamma, m\delta + n\beta + \beta\delta >$ for $\widetilde{M} > 0,\ \widetilde{N} > 0$.

(5) Inverse: The inverse of a TFN $\widetilde{M} = < m, \alpha, \beta >, m > 0$ is defined as, $\widetilde{M}^{-1} = < m, \alpha, \beta >^{-1} \approx < m^{-1}, \beta m^{-2}, \alpha m^{-2} >$.

This is also an approximate value of $\widetilde{M}^{-1}$ and it is valid only a neighbourhood of $1/m$. Division of $\widetilde{M}$ by $\widetilde{N}$ is given by,

$$\frac{\widetilde{M}}{\widetilde{N}} = \widetilde{M}.\widetilde{N}^{-1}$$

Since inverse and product both are approximate, the division is also an approximate value. The formal definition of division is given below.

(6) Division:

$$\frac{\widetilde{M}}{\widetilde{N}} = \widetilde{M}.\widetilde{N}^{-1} = < m, \alpha, \beta >. < n^{-1}, \delta n^{-2}, \gamma n^{-2} > \approx < \frac{m}{n}, \frac{m\delta + n\alpha}{n^2}, \frac{m\gamma + n\beta}{n^2} >$$

From the definition of multiplication ofTFNs, the power of any TFN $\widetilde{M}$is defined in the following way.

(7) Exponentiation : Using the definition of multiplication it can be shown that $\widetilde{M}^n$is given by

$\widetilde{M}^n = < m, \alpha, \beta >^n \approx < m^n, -nm^{n-1}\beta, -nm^{n-1}\alpha >$, when n is negative ,

$\approx < m^n, nm^{n-1}\alpha, -nm^{n-1}\beta >$, when n is positive.

Consider two TFN's with a common mean value. Then subtraction produces a TFN whose mean value is zero and the spreads are the sum of both the spreads of computed TFN. The quotient of same TFNs is a TFN having mean value one. THE Inverse of a TFN whose mean value is zero does not exist and we cannot divide by such a number. The addition and multiplication of TFNs are both commutative and associative. But the distributive law does not always hold.

For example, if $\tilde{A} = < 2,0.5,0.5 >, \tilde{B} = < 3,0.8,0.7 >, \tilde{C} = < 5,1,2 >$and $\widetilde{D} = < -5,2,1 >$, then $\tilde{A}.(\tilde{B} + \tilde{C}) = \tilde{A}.\tilde{B} + \tilde{A}.\tilde{C}$ holds but, $\tilde{A}.(\tilde{C} + \widetilde{D}) \neq \tilde{A}.\tilde{C} + \tilde{A}.\widetilde{D}$.

It may be remembered that

$$< m, \alpha, \beta >. < 0,0,0 > = < 0,0,0 >.$$

4. Triangular fuzzy matrix

Definition 4.1. Triangular fuzzy matrix (TFM): A triangular fuzzy matrix of order $m \times n$ is defined as $\tilde{A} = (a_{ij})_{m\times n}$, where $a_{ij} = <m_{ij}, \alpha_{ij}, \beta_{ij}>$ is the ijth element of $\widetilde{A}$, $m_{ij}$is the mean value of a_{ij} and α_{ij}, β_{ij} are the left and right spreads of a_{ij} respectively.

As for classical matrices define the following operations on TFMs. Let $\widetilde{A} = (a_{ij})$ and$\widetilde{B} = (b_{ij})$ be two TFMs of same order. Then we have the following :

$(i)\ \tilde{A} + \tilde{B} = (a_{ij} + b_{ij})$ $(ii)\ \tilde{A} - \tilde{B} = (a_{ij} - b_{ij})$,

(iii) For $\tilde{A} = (a_{ij})_{m\times n}$ and $\tilde{B} = (b_{ij})_{n\times p}$, $\tilde{A}.\tilde{B} = (c_{ij})_{m\times p}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}.\ b_{kj}$, $i = 1,2,\ldots,m$ and $j = 1,2,..,p$.
$(iv)\tilde{A}' = (a_{ji})$ (the transpose of $\tilde{A}$)
(v) $k.\tilde{A} = (ka_{ij})$, where k is a scalar.

We now define special types of TFMs corresponding to special classical matrices. However, because of fuzziness we will have more than one type of TFM corresponding to one type of classical matrix.

Definition 4.2. Pure null TFM: A TFM is said to be a pure null TFM if all its entries are zero, i.e., all elements are $< 0,0,0 >$. This matrix is denoted by O.

Definition 4.3. Fuzzy null TFM: A TFM is said to be a fuzzy null TFM if all elements are of the form $a_{ij} =< 0, \varepsilon_1, \varepsilon_2 >$ where $\varepsilon_1.\varepsilon_2 \neq 0$.

Definition 4.4. Pure unit TFM: A square TFM is said to be a pure unit TFM if $a_{ii} =< 1,0,0 >$ and $a_{ij} =< 0,0,0 >, i \neq j$, for all i, j. It is denoted by $\tilde{I}$.

Definition 4.5. Fuzzy unit TFM: A square TFM is said to be a fuzzy unit TFM if $a_{ii} = < 1, \varepsilon_1, \varepsilon_2 >$ and $a_{ij} = < 0, \varepsilon_3, \varepsilon_4 >$ for $i \neq j$ for all i, j, where $\varepsilon_1.\varepsilon_2 \neq 0$, $\varepsilon_3.\varepsilon_4 \neq 0$.

Definition 4.6. Pure triangular TFM: A square TFM $\tilde{A} = (a_{ij})$ is said to be a pure triangular TFM if either $a_{ij} =< 0,0,0 >$ for all $i > j$ or $a_{ij} =< 0,0,0 >$ for all $i < j$; $i, j = 1,2,\cdots,n$.

A pure triangular TFM $\tilde{A} = (a_{ij})$ is said to be pure upper triangular TFM when $a_{ij} = < 0,0,0 >$ for all $i > j$ and is said to be a pure lower triangular TFM if $a_{ij} =< 0,0,0 >$ for all $i < j$.

Definition 4.7. Fuzzy triangular TFM: A square TFM $\tilde{A} = (a_{ij})$ is said to be a fuzzy triangular TFM if either $a_{ij} =<0,\varepsilon_1, \varepsilon_2 >$ for all $i > j$ or $a_{ij} =< 0, \varepsilon_1, \varepsilon_2 >$for all $i < j$; $i, j = 1,2,\cdots,n$ and $\varepsilon_1.\varepsilon_2 \neq 0$.

Definition 4.8. Symmetric TFM: A square TFM $\tilde{A} = (a_{ij})$ is said to be symmetric if $\tilde{A} = \tilde{A}'$, i.e., if $a_{ij} = a_{ji}$ for all i, j.

Definition 4.9. Pure skew-symmetric TFM: A square TFM $\tilde{A} = (a_{ij})$ is said to be pure skew-symmetric if $\tilde{A} = -\tilde{A}'$ and $a_{ii} = < 0,0,0 >$, i.e., if $a_{ij} = -a_{ji}$ for all i, j and $a_{ii} = < 0,0,0 >$

Definition 4.10. Fuzzy Skew-symmetric TFM: A square TFM $\tilde{A} = (a_{ij})$ is said to be fuzzy skew-symmetric if $\tilde{A} = -\tilde{A}'$ and $a_{ii} = <0,\varepsilon_1, \varepsilon_2 >$ i.e., if $a_{ij} = -a_{ji}$ for all i, j and $a_{ii} = < 0, \varepsilon_1, \varepsilon_2 >, \varepsilon_1.\varepsilon_2 \neq 0$

5. Adjoint and determinant of triangular fuzzy matrix

The triangular fuzzy determinant (TFD) of a TFM, minor and cofactor are defined as in classical matrices. But, TFD has some special properties due to the sub-distributive property of TFNs.

Definition 5.1. Determinant of TFM: The triangular fuzzy determinant of a TFM $\tilde{A}$of order $n \times n$ is denoted by $|\tilde{A}|$ or $det(\tilde{A})$ and is defined as,

$$|\tilde{A}| = \sum_{\sigma \epsilon S_n} Sgn\, \sigma < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} > \ldots\ldots < m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >$$
$$= \sum_{\sigma \epsilon S_n} Sgn\, \sigma \prod_{i=1}^{n} a_{i\sigma(i)} ,$$

where $a_{i\sigma(i)} =< m_{i\sigma(i)}, \alpha_{i\sigma(i)}, \beta_{i\sigma(i)} >$ are TFNs and S_n denotes the symmetric group of all permutations of the indices { 1,2,....,n } and $Sgn\, \sigma = 1$ or -1 according as the permutation $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$ is even or odd respectively.

The computation of $det(\tilde{A})$ involves several product of TFNs. Since the product of two or more TFNs is an approximate TFN, the value of $det(\tilde{A})$ is also an approximate TFN.

Definition 5.2. Minor: Let$\tilde{A} = (a_{ij})$ be a square TFM of order$n \times n$. The minor of an element a_{ij} in $det(\tilde{A})$ is a determinant of order $(n-1) \times (n-1)$, which is obtained by deleting the ith row and the jth column from $\tilde{A}$ and is denoted by $\widetilde{M}_{ij}$.

Definition 5.3. Cofactor: Let $\tilde{A} = (a_{ij})$ be a square TFM of order $n \times n$. The cofactor of an element a_{ij} in $\tilde{A}$ is denoted by and $\tilde{A}_{ij}$ is defined as $\widetilde{A}_{ij} = (-1)^{i+j} \widetilde{M}_{ij}$.

Definition 5.4. Adjoint: Let $\widetilde{A} = (a_{ij})$ be a square TFM and $\tilde{B} = (A_{ij})$ be a square TFM whose elements are the cofactors of the corresponding elements in $|\tilde{A}|$ then the transpose of $\tilde{B}$is called the adjoint or adjugate of $\widetilde{A}$ and it is equal to $(\tilde{A}_{ji})$. The adjoint of $\widetilde{A}$ is denoted by $adj(\tilde{A})$.

Here $|\tilde{A}|$ contains $n!$ terms out of which $\frac{n}{2}!$ are positive terms and the same number of terms are negative. All these $n!$ terms contain n quantities at a time in product form, subject to the condition that from the n quantities in the product exactly one is taken from each row and exactly one from each column.

Alternatively, a TFD of a TFM $\widetilde{A} = (a_{ij})$ may be expanded in the form $\sum_{j=1}^{n} a_{ij}.A_{ij},\ i \epsilon \{1,2,\ldots,n\}$, where $\tilde{A}_{ij}$ is the cofactor of a_{ij}. Thus the TFD is the sum of the products of the elements of any row (or column) and the cofactors of the corresponding elements of the same row (or column). We refer to this method as the alternative method.

In classical mathematics, the value of a determinant is computed by any one of the aforesaid two processes and both yield same result. But, due to the failure of distributive laws of triangular fuzzy numbers, the value of a TFD, computed by the aforesaid two processes will differ from each other. For this reason the value of a TFD should be determined according to the definition, i.e., using the following rule only

$$|\tilde{A}| = \sum_{\sigma \epsilon S_n} Sgn\, \sigma < m_{1\sigma(1)}, \alpha_{1\sigma(1)}, \beta_{1\sigma(1)} > \cdots < m_{n\sigma(n)}, \alpha_{n\sigma(n)}, \beta_{n\sigma(n)} >.$$

On the other hand the value of a TFD computed by the alternative process yields a different and less desirable result.

Theorem 5.5. Let $\widetilde{A}$ be a square TFM of order n. Then $\tilde{A}.adj\tilde{A} = adj\,\tilde{A}.\tilde{A} = (det\,\tilde{A})\tilde{I}_n$.

Proof: Let $\widetilde{A}$ be a square TFM of order n is defined as $\tilde{A} = (a_{ij})_{n\times n}$ where $a_{ij} = < m_{ij}, \alpha_{ij}, \beta_{ij} >$ is the ijth element of $\tilde{A}$. m_{ij} is the mean value of a_{ij} and α_{ij}, β_{ij} are the left and right spreads respectively.

Let, $\widetilde{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

Then $adj\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{21} & \dots & \tilde{A}_{n1} \\ \tilde{A}_{12} & \tilde{A}_{22} & \dots & \tilde{A}_{n2} \\ \dots & \dots & \dots & \dots \\ \tilde{A}_{1n} & \tilde{A}_{2n} & \dots & \tilde{A}_{nn} \end{pmatrix}$, where $\tilde{A}_{ij}$ is the cofactor, of $a_{ij} = < m_{ij}, \alpha_{ij}, \beta_{ij} >$ in $det(\tilde{A})$.

i.e. $\tilde{A}_{11} = \begin{vmatrix} < m_{22}, \alpha_{22}, \beta_{22} > & < m_{23}, \alpha_{23}, \beta_{23} > & \dots & < m_{2n}, \alpha_{2n}, \beta_{2n} > \\ < m_{32}, \alpha_{32}, \beta_{32} > & < m_{33}, \alpha_{33}, \beta_{33} > & \dots & < m_{3n}, \alpha_{3n}, \beta_{3n} > \\ \dots & \dots & \dots & \dots \\ < m_{n2}, \alpha_{n2}, \beta_{n2} > & < m_{n3}, \alpha_{n3}, \beta_{n3} > & \dots & < m_{nn}, \alpha_{nn}, \beta_{nn} > \end{vmatrix}$

Now, $\tilde{A}.adj\tilde{A} = \begin{pmatrix} \sum_{k=1}^{n} a_{1k}\tilde{A}_{1k} & \sum_{k=1}^{n} a_{1k}\tilde{A}_{2k} & \dots & \sum_{k=1}^{n} a_{1k}\tilde{A}_{nk} \\ \sum_{k=1}^{n} a_{2k}\tilde{A}_{1k} & \sum_{k=1}^{n} a_{2k}\tilde{A}_{2k} & \dots & \sum_{k=1}^{n} a_{2k}\tilde{A}_{nk} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \sum_{k=1}^{n} a_{nk}\tilde{A}_{1k} & \sum_{k=1}^{n} a_{nk}\tilde{A}_{2k} & \dots & \sum_{k=1}^{n} a_{nk}\tilde{A}_{nk} \end{pmatrix}$

$$= \begin{pmatrix} \det\tilde{A} & \tilde{0} & \dots & \tilde{0} \\ \tilde{0} & det\,\tilde{A} & \dots & \tilde{0} \\ \dots & \dots & \dots & \dots \\ \tilde{0} & \tilde{0} & \dots & det\,\tilde{A} \end{pmatrix} = (det\tilde{A})\tilde{I}_{n},$$

Since $\sum_{k=1}^{n} a_{ik}\tilde{A}_{jk} = det\tilde{A},\ if\ i = j$

$= \tilde{0},\ if\ i \neq j$ where $\widetilde{0} =< 0, \varepsilon_1, \varepsilon_2 >$ is a zero TFN, $\varepsilon_1.\varepsilon_2 \neq 0$

Again $adj\tilde{A}.\tilde{A} = \begin{pmatrix} \sum_{k=1}^{n} \tilde{A}_{k1}a_{k1} & \sum_{k=1}^{n} \tilde{A}_{k1}a_{k2} & \dots & \sum_{k=1}^{n} \tilde{A}_{k1}a_{kn} \\ \sum_{k=1}^{n} \tilde{A}_{k2}a_{k1} & \sum_{k=1}^{n} \tilde{A}_{k2}a_{k2} & \dots & \sum_{k=1}^{n} \tilde{A}_{k2}a_{kn} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \sum_{k=1}^{n} \tilde{A}_{kn}a_{k1} & \sum_{k=1}^{n} \tilde{A}_{kn}a_{k2} & \dots & \sum_{k=1}^{n} \tilde{A}_{kn}a_{kn} \end{pmatrix}$

$$= \begin{pmatrix} \det\tilde{A} & \tilde{0} & \dots & \tilde{0} \\ \tilde{0} & det\,\tilde{A} & \dots & \tilde{0} \\ \dots & \dots & \dots & \dots \\ \tilde{0} & \tilde{0} & \dots & det\,\tilde{A} \end{pmatrix} = (det\tilde{A}\,)\tilde{I}_{n},$$

Since $\sum_{k=1}^{n} a_{ki}\,\tilde{A}_{kj} = det\tilde{A},\ if\ i = j$

$= \tilde{0}, if i \neq j$ where $\widetilde{0} = < 0, \varepsilon_1, \varepsilon_2 >$ is a zero equivalent TFN,$\varepsilon_1 . \varepsilon_2 \neq 0$

And $\tilde{I}_n$ is the fuzzy unit TFM where $a_{ii} =< 1, \varepsilon_1, \varepsilon_2 >$ and$a_{ij} =< 0, \varepsilon_3, \varepsilon_4 >$ for$i \neq j$ for all i, j and $\varepsilon_1 . \varepsilon_2 \neq 0$, $\varepsilon_3 . \varepsilon_4 \neq 0$

Therefore, $\widetilde{A}. adj\tilde{A} = adj\tilde{A}.\tilde{A} = \left(det\tilde{A}\right)\tilde{I}_n$.

6. Inverse of triangular fuzzy matrix

Definition 6.1. Singular TFM: Let $\tilde{A} = (a_{ij})$ be a square TFM of order$n$,then it is said to be Singular TFM if $det(\tilde{A})= \tilde{0}$. where $\widetilde{0} = < 0, \varepsilon_1, \varepsilon_2 >$ is a zero equivalent TFN,$\varepsilon_1 . \varepsilon_2 \neq 0$.

Definition 6.2. Non-Singular TFM: Let $\tilde{A} =(a_{ij})$ be asquare TFM of order n, then it is said to be Non-Singular TFM if $det\left(\tilde{A}\right) \neq \widetilde{0}$.

Definition 6.3. Inverse of a TFM: A non-singular square TFM $\widetilde{A}$ of order n is said to be invertible if there exist a TFM $\tilde{B}$ such that $\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = \tilde{I}_n$. $\tilde{B}$ is said to be an inverse of $\tilde{A}$ and is denoted by $\tilde{A}^{-1}$ .Thus $\tilde{A}\tilde{A}^{-1} = \tilde{I}_n = \tilde{A}^{-1}\tilde{A}$. Also $\tilde{A}^{-1} = \frac{1}{\det \tilde{A}} adj\tilde{A}$.

In order that both $\tilde{A}\widetilde{B}$ and $\widetilde{B}\tilde{A}$ should exist, $\widetilde{B}$ must be a square TFM of order n.

6.4. Numerical example

Evaluate the inverse of $\widetilde{A} = \begin{pmatrix} < 3, 0.5, 0.5 > & < 2, 0.2, 0.2 > \\ < 2, 0.1, 0.1 > & < 2, 0.5, 0.5 > \end{pmatrix}$

Solution: $det\left(\tilde{A}\right) = < 3, 0.5, 0.5 >< 2, 0.5,0.5 > - < 2, 0.2, 0.2 >< 2, 0.1, 0.1 >$

$=< 6, 2.5, 2.5 > - < 4, 0.6, 0.6 >$

$=< 2, 3.1, 3.1 >$

Now, $adj(\tilde{A}) = \begin{pmatrix} < 2, 0.5, 0.5 > & < -2, 0.2, 0.2 > \\ < -2, 0.1, 0.1 > & < 3, 0.5, 0.5 > \end{pmatrix}$

Then from the definition of inverse we have,

$$\tilde{A}^{-1} = \frac{1}{\det(\tilde{A})} adj(\tilde{A})$$

$$= \begin{pmatrix} < 1, 1.8, 1.8 > & < -1, -1.45, -1.45 > \\ < -1, -1.5, -1.5 > & < 1.5, 2.575, 2.575 > \end{pmatrix}$$

Check: Now $\tilde{A}.\tilde{A}^{-1} =$

$$\begin{pmatrix} < 3, 0.5, 0.5 > & < 2, 0.2, 0.2 > \\ < 2, 0.1, 0.1 > & < 2, 0.5, 0.5 > \end{pmatrix} \begin{pmatrix} < 1, 1.8, 1.8 > & < -1, -1.45, -1.45 > \\ < -1, -1.5, -1.5 > & < 1.5, 2.575, 2.575 > \end{pmatrix}$$

$$= \begin{pmatrix} < 3, 5.9, 5.9 > + < -2, -2.8, -2.8 > & < -3, -3.85, -3.85 > + < 3, 5.45, 5.45 > \\ < 2, 3.7, 3.7 > + < -2, -2.5, -2.5 > & < -2, -2.8, -2.8 > + < 3, 5.9, 5.9 > \end{pmatrix}$$

$$= \begin{pmatrix} < 1, 3.1, 3.1 > & < 0, 1.6, 1.6 > \\ < 0, 1.2, 1.2 > & < 1, 3.1, 3.1 > \end{pmatrix} = \tilde{I}_2 = \tilde{A}^{-1}.\tilde{A}$$

Definition 6.5. Defuzzification: We define a function $\widetilde{D} : F(R) \rightarrow R$ which maps each fuzzy numbers to real line. $F(R)$ represents the set of all Triangular Fuzzy Numbers.If$\widetilde{D}$ be any linear defuzzification functions then ,

$$\widetilde{D}(\widetilde{M}) = \left(\frac{4m+\alpha+\beta}{6}\right) \text{ where } \widetilde{M} = <m, \alpha, \beta> \text{ be any TFN.}$$

We can also evaluate the inverse of a TFM by using defuzzified value of $det(A)$ instead of (A) . i.e. $\tilde{A}^{-1} = \frac{1}{\widetilde{D}(det(\tilde{A}))} adj(\tilde{A})$.

6.6. Numerical example

Evaluate the inverse of $\tilde{A} = \begin{pmatrix} <3, 0.5, 0.5> & <2, 0.2, 0.2> \\ <2, 0.1, 0.1> & <2, 0.5, 0.5> \end{pmatrix}$ using defuzzified value of $det(\tilde{A})$.

Solution:

We have already, $det(\tilde{A}) = <2, 3.1, 3.1>$

Now, $\widetilde{D}(det(\tilde{A})) = \frac{8+3.1+3.1}{6} = 2.37$

We have $adj(\tilde{A})$= $\begin{pmatrix} <2, 0.5, 0.5> & <-2, 0.2, 0.2> \\ <-2, 0.1, 0.1> & <3, 0.5, 0.5> \end{pmatrix}$

Then, $\widetilde{\ }A^{-1} = \frac{1}{\widetilde{D}(det(\tilde{A}))} adj(\tilde{A})$

$$= \frac{1}{2.37}\begin{pmatrix} <2, 0.5, 0.5> & <-2, 0.2, 0.2> \\ <-2, 0.1, 0.1> & <3, 0.5, 0.5> \end{pmatrix}$$

$$= \begin{pmatrix} <0.84, 0.21, 0.21> & <-0.84, 0.084, 0.084> \\ <-0.84, 0.042, 0.042> & <1.26, 0.21, 0.21> \end{pmatrix}$$

Check: $\widetilde{A}.\tilde{A}^{-1} =$

$$\begin{pmatrix} <3, 0.5, 0.5> & <2, 0.2, 0.2> \\ <2, 0.1, 0.1> & <2, 0.5, 0.5> \end{pmatrix}\begin{pmatrix} <0.84, 0.21, 0.21> & <-0.84, 0.084, 0.084> \\ <-0.84, 0.042, 0.042> & <1.26, 0.21, 0.21> \end{pmatrix}$$

$$= \begin{pmatrix} <0.98, 1.302, 1.302> & <0, 1.344, 1.344> \\ <0, 1.008, 1.008> & <0.98, 1.302, 1.302> \end{pmatrix} \approx \tilde{I}_2 = \tilde{A}^{-1}.\tilde{A}$$

7. Matrix inversion method for fuzzy linear system on TFN

In this section, we define the concept of fuzzy linear system is justify in matrix inversion method with the aid of Triangular Fuzzy Numbers and the relevant definitions are recalled in nature.

Consider the system of n fuzzy linear non- homogeneous TFN equations in n unknown TFN vectors $x_1, x_2, \dots, x_n$.

Here a_{ij} , x_i , b_i are triangular fuzzy numbers. For all $i , j = 1,2, \dots, n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

The above linear system is represented in the form is given by,

$$\tilde{A}X = \tilde{b}$$

where $\widetilde{A} = (a_{ij})$, $1 \leq i,j \leq n$ is Triangular Fuzzy Matrix of order n and $a_{ij} \in F(R)$ and $(x_i, b_i) \in F(R)$, for all $i = 1,2, \dots, n$ and $j = 1,2, \dots, n$ .This system is called Fuzzy Linear System (FLS).

If the coefficient Triangular Fuzzy matrix $\tilde{A}$ is non singular, then we get ,

$$\tilde{A}^{-1}(\tilde{A}X) = \tilde{A}^{-1}\tilde{b}$$
$$(\tilde{A}^{-1}\tilde{A})X = \tilde{A}^{-1}\tilde{b}$$

$$X = \tilde{A}^{-1}\tilde{b}$$

The solution of FLS will be represented by,

$$X = \tilde{A}^{-1}\tilde{b}$$

7.1. Numerical example

In this section a simple example is given in order to illustrate the two proposed method.

Consider the following fuzzy linear system and solve by matrix inversion method .

$$< 3, 0.5, 0.5 > x_1 + < 2, 0.2, 0.2 > x_2 = < 5, 0.1, 0.1 >$$
$$< 2, 0.1, 0.1 > x_1 + < 2, 0.5, 0.5 > x_2 = < 4, 0.1, 0.1 >$$

Solution:

The given linear system may be written as,

$$\begin{pmatrix} < 3, 0.5, 0.5 > & < 2, 0.2, 0.2 > \\ < 2, 0.1, 0.1 > & < 2, 0.5, 0.5 > \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} < 5, 0.1, 0.1 > \\ < 4, 0.1, 0.1 > \end{pmatrix}$$
$$X = \tilde{A}^{-1}\tilde{b}$$

$\boldsymbol{i}$) **Using determinant of $\widetilde{\boldsymbol{A}}$:**

Now $det(\tilde{A}) = < 2, 3.1, 3.1 >$

Since $det(\tilde{A}) \neq \tilde{0}$ where $\tilde{0} = < 0, \varepsilon_1, \varepsilon_2 >$ s.t. $\varepsilon_1 . \varepsilon_2 \neq 0$

So, $\tilde{A}$ is non-singular, then $\tilde{A}^{-1}$exists. $\tilde{A}^{-1} = \frac{1}{\det(\tilde{A})} adj(\tilde{A})$

$$= \begin{pmatrix} < 1, 1.8, 1.8 > & < -1, -1.45, -1.45 > \\ < -1, -1.5, -1.5 > & < 1.5, 2.575, 2.575 > \end{pmatrix}$$

The solution is $X = \tilde{A}^{-1}\tilde{b}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} < 1, 1.8, 1.8 > & < -1, -1.45, -1.45 > \\ < -1, -1.5, -1.5 > & < 1.5, 2.575, 2.575 > \end{pmatrix} \begin{pmatrix} < 5, 0.1, 0.1 > \\ < 4, 0.1, 0.1 > \end{pmatrix}$$
$$= \begin{pmatrix} < 5, 9.1, 9.1 > + < -4, -5.7, -5.7 > \\ < -5, -7.4, -7.4 > + < 6, 10.45, 10.45 > \end{pmatrix} = \begin{pmatrix} < 1, 3.4, 3.4 > \\ < 1, 3.05, 3.05 > \end{pmatrix}$$

The solution is $x_1 = < 1, 3.4, 3.4 >$ $x_2 = < 1, 3.05, 3.05 >$.

$\boldsymbol{ii}$) **Using defuzzied value**

Now, $\widetilde{D}(det(\tilde{A})) = \frac{8+3.1+3.1}{6} = 2.37$

Since $\widetilde{D}(det(\tilde{A})) \neq 0$

So, $\tilde{A}$ is non-singular and $\tilde{A}^{-1}$ exists.

Then $\tilde{A}^{-1} = \frac{1}{\widetilde{D}(det(\tilde{A}))} adj(\tilde{A}) =$

$$\begin{pmatrix} < 0.84, 0.21, 0.21 > & < -0.84, 0.084, 0.084 > \\ < -0.84, 0.042, 0.042 > & < 1.26, 0.21, 0.21 > \end{pmatrix}$$

The solution is $X = \tilde{A}^{-1}\tilde{b}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} < 0.84, 0.21, 0.21 > & < -0.84, 0.084, 0.084 > \\ < -0.84, 0.042, 0.042 > & < 1.26, 0.21, 0.21 > \end{pmatrix} \begin{pmatrix} < 5, 0.1, 0.1 > \\ < 4, 0.1, 0.1 > \end{pmatrix}$$
$$= \begin{pmatrix} < 4.2, 1.134, 1.134 > + < -3.36, 0.42, 0.42 > \\ < -4.2, 0.294, 0.294 > + < 5.04, 0.966, 0.966 > \end{pmatrix}$$
$$= \begin{pmatrix} < 0.84, 1.76, 1.76 > \\ < 0.84, 1.26, 1.26 > \end{pmatrix}$$

The solution is $x_1 = < 0.84, 1.76, 1.76 >$ $x_2 = < 0.84, 1.26, 1.26 >$

Calculation of residues: These two solutions are slightly different. Now we will calculate the sum of square of residues S_1^2 , S_2^2 of method i) and ii) respectively.

$$S_1^2 = [< 3,0.5,0.5 >< 1,3.4,3.4 > + < 2,0.2,0.2 >< 1,3.05,3.05 > - < 5,0.1,0.1 >]^2$$
$$+[< 2,0.1,0.1 >< 1,3.4,3.4 > + < 2,0.5,0.5 >< 1,3.05,3.05 > - < 4,0.1,0.1 >]^2$$
$$= [< 0, 17.1, 17.1 >]^2 + [< 0, 13.6, 13.6 >]^2 = < 0,0,0 >$$

$$S_2^2 =$$
$$[< 3,0.5,0.5 >< 0.84,1.76,1.76 > + < 2,0.2,0.2 >< 0.84,1.26,1.26 > - < 5,0.1,0. >]^2$$
$$+[< 2,0.1,0.1 >< 0.84,1.76,1.76 > + < 2,0.5,0.5 >< .84,1.26,1.26 > - < 4,0.1,0 >]^2$$
$$= [< -0.8,8.488,8.488 >]^2 + [< -0.64,6.644,6.644 >]^2 = < 0.409,8.504,8.504 >$$

Since S_1^2 is exactly $< 0,0,0 >$ so, the solution obtained by using the determinant value of Agives exact solution.

8. Conclusion

In this article, some elementary operations on triangular fuzzy numbers are defined. Like classical matrices we also define some operations on TFMs. Using the elementary operations, some important properties of TFMs are presented. The concept of adjoint of TFM is discussed and some properties on it are also presented. The definition and some properties of determinant of TFM are presented in this article. It is well known that the determinant is a very important tool in mathematics, so an efficient method is required to evaluate a TFD. Presently, we are trying to develop an efficient method to evaluate a TFD of large size. Some special types of TFMs, i.e. pure and fuzzy triangular, symmetric, pure and fuzzy skew-symmetric, singular, semi-singular and constant TFMs are defined here. Then we evaluate the inverse of a TFM using two methods. One is general adjoint determinant method. Other is defuzzified method. Then we investigated the fuzzy linear system of equations with fuzzy coefficients involving in fuzzy variables. The matrix inversion method is used to solve the system of equations. This method is illustrated with numerical example and the inverse of the corresponding coefficient matrix is obtained using both methods. The solution obtained using determinant value of coefficient matrix is most exact. The notion of FLS can be applying in Cramer's rule and LU decomposition method by this proposed method in future.

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