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Progress in Nonlinear Dynamics and Chaos

Convergence, Permanent and g-inverse of m-Polar Fuzzy Matrix

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Abstract. In this paper, m-polar fuzzy matrix is introduced. In fuzzy matrix, each element represents the membership value of an element, while in m-polar fuzzy matrix each element is a vector containing m elements and value of each element lies between 0 and 1 including 0 and 1. The convergence, permanent and g-inverse of m-polar fuzzy matrix are defined and explained with examples. Some useful properties are also presented.

Keywords: fuzzy matrix, m-polar fuzzy matrix, convergence of fuzzy matrix, permanent, g-inverse of fuzzy matrix

AMS Mathematics Subject Classification (2010): 15B15

1. Introduction

Fuzzy matrices were introduced for the first time by Thomason [4], who discussed the convergence of powers of fuzzy matrix. Xin studied the controllable fuzzy matrix. Ragab et al. [3] presented some properties of the min-max composition of fuzzy matrices. Kim et al. [2] presented some important results on determinant of a square fuzzy matrices.

Several other types of matrices are available on fuzzy setup. There are some limitations in dealing with uncertainties by fuzzy set. Pal et al. defined intuitionistic fuzzy determinant in 2001 [18] and intuitionistic fuzzy matrices (IFMs) in 2002 [19]. Bhowmik and Pal [8] introduced some results on IFMs, intuitionistic circulant fuzzy matrix and generalized intuitionistic fuzzy matrix [8-14]. Shyamal and Pal [25-27] defined the distances between IFMs and hence defined a metric on IFMs. They also cited few applications of IFMs. In [17], the similarity relations, invertibility conditions and eigenvalues of IFMs are studied. Idempotent, regularity, permutation matrix and spectral radius of IFMs are also discussed. The parameterizations tool of IFM enhances the flexibility of its applications. For other works on IFMs see [5-7,16,22,23,26,27]. The concept of interval-valued fuzzy matrices (IVFMs) as a generalization of fuzzy matrix was introduced and developed in 2006 by Shaymal and Pal [28] by extending the max-min operation in fuzzy algebra. For more works on IVFMs see [21]. Combining IFMs and IVFMs, a new fuzzy matrix called interval-valued intuitionistic fuzzy matrices (IVIFMs) is defined [15]. For other works on IVIFMs, see [12,14]. For recent works on uncertain matrix theory see [30-33]. The m-polar fuzzy set is also investigated in many other areas such as m-polar fuzzy graph theory [34-39].

In this paper, the m-polar fuzzy matrix is introduced and convergence, permanent and g-inverse are investigated with suitable examples. Many results are also presented here.

2. Convergence of m-polar fuzzy matrix

The m-polar fuzzy matrix (mFM) is a generalization of fuzzy matrix in terms of complements of elements. In fuzzy matrix, each element represents the membership value of an element and its values lies between 0 and 1 including 0 and 1. On the other hand in m-polar fuzzy matrix each element is a vector containing m elements and value of each element lies between 0 and 1 including 0 and 1. An example of mFM is shown below:

$$\begin{pmatrix} (0.2,0.5,0.6) & (0.3,0.2,0.1) \\ (0.6,0.9,0.2) & (0.4,0.5,0.3) \\ (0.4,0.2,0.8) & (0.5,0.4,0.7) \end{pmatrix}$$

This is a 3×3 matrix and each element has three components. So, it is a 3FM of order 3×3.

The set of all mFMs of order $m \times n$ is denoted by M_{mn} and that of order $n \times n$ is denoted by M_n.

In this section, we introduce the concept of convergence and power of convergence of mFM. In general, we know that, the sequence of matrices $A_1,A_2,A_3,\ldots,A_{m+1},\ldots$,That is $\{A_m\}$ is said to be converge to a finite matrix A (if exists) if

$$\lim_{m\to\infty} A_m = A.$$

Definition 2.1. (Power convergence of mFM)
A least positive integer p is said to the power of convergence of a mFM A in respectively to a binary composition * if

$$A^{p+n}=A^{p+n-1}=A^{p+n-2}=\ldots\ldots.=A^{p+1}=A^p$$

where, n∈N (set off all natural number) and

$$A^2 = A * A, A^3 = A * A * A = A^2 * A, \text{ and so on.}$$

The number p is called the index of A and is denoted by i(A).

Definition 2.2. The partial order relation '≤' over M_m is defined as $A \le B$ iff $a_{ij} \le b_{ij}$ for all $i, j \in \{1,2,3,\ldots\ldots,m\}$ where $A=(a_{ij})$, $B=(b_{ij})\in M_m$. where $a_{ij}=(a_{ij}^1,a_{ij}^2, \ldots..,a_{ij}^m)$; $b_{ij}=(b_{ij}^1,b_{ij}^2,\ldots\ldots,b_{ij}^m)$, that is, $A \le B$ iff $A+B=B$, $A<B$ holds iff $A\le B$ and $A\neq B$.

Definition 2.3. A matrix a is said to be **nilpotent** of order k if $A^k = 0$ for some k positive integer, and A is **idempotent** if $A^2=A$.

Lemma 2.1. Let $A=(a_{ij})\in \mu_m$be a mFM. If r > m, then $A^r \le \sum_{k=0}^{m-1} A^k$, where $A^0=I_m$.
As a result, $A^{r+1} \le \sum_{k=1}^{m} A^k$.
Proof: Let $B=\sum_{k=0}^{m-1} A^k$. Now, $a_{ii}^{(r)} \le i_b = b_{ii}$. since, $a_{ii}^{(0)} = i_b$.

If $i \neq j$, we consider, an arbitrary summand of RHS of equality, i.e. $a_{ij1},a_{i1j2},a_{i2j3},\ldots\ldots,a_{ir-1j}$ Since $j_1,j_2,\ldots..,j_{r-1}$, $j\in\{1,2,3,\ldots..,m\}$ and $r+1 > m$ there are such that such that j_s-j_n ($0 \leq s < t \leq r$, $j_0 = I$, $j_m = j$).

Deleting $a_{js}\ j_{s+1},a_{js+1}j_{s+2},\ldots..,a_{jt-1}j_t$ from the summand $a_{ij1},a_{j1j2},a_{j2j3},\ldots..,a_{jr-1j,}$ we obtain, $a_{ij1},a_{j1j2},a_{j2j3},\ldots,a_{jr-1j}\leq a_{ij1},a_{j1j2},a_{j2j3},\ldots..,a_{js-1j,}a_{jr-1j}$

If the number s+r-t+2 of the subscript in the RHS of the above inequality still more than m, the same deleting method is used.

Therefore, there is a positive integer q≤m-1such that $a_{ij_1,}a_{j_1j_{2,}}a_{j2j_3,\ldots\ldots,}a_{j_{r-1}}j \leq a_{il_1}, a_{l_1{}_{l_2}}, a_{l_2{}_{l_3}},\ldots\ldots, a_{l_{q-1}j}$. Hence by definition of A^r we have, $A_{ij}{}^r\leq\sum_{k=1}^{m-1} a_{ij}{}^{(k)} = b_{ij}$

i.e. $A^r\leq\sum_{k=1}^{m-1} A^k \leq \sum_{k=0}^{m-1} A^k$.

Definition 2.4. Let A, B, C $\in \mu_m$, the m-polar fuzzy matrix A is said to be **transitive**, if $A^2 \leq A$. the mFM B is said to be transitive closure of matrix A, if B is transitive, $A \leq B$ and $B \leq C$ for any transitive matrix C, satisfying $A \leq C$, the transitive closure of A is denoted by t(A).

Theorem 2.1. Let $A\in\mu_m$ be mFM. Then the transitive closure of A is given by $t(A)=\sum_{k=1}^{m} A^k$.

Proof: Let, $B=\sum_{k=1}^{m} A^k$, obviously $A \leq B$ since μ_m is idempotent under addition, we have $B^2=\sum_{k=2}^{2m} A^k \leq \sum_{k=1}^{2m} A^k$

Or, $B^2\leq B+\sum_{K=m+1}^{2m} A^k$

By Lemma 2.1,

$A^k\leq \sum_{t=1}^{m} A^t = B$ as $k > m$

Hence, $B^2 \leq B$.

If there is a matrix C such that $A \leq C$ and $C^2 \leq C$.

Then $A^2 \leq AC \leq C^2 \leq C$ and by induction, we have $A^K \leq C^K \leq C$ for all positive integer k hence, $B \leq C$.

Thus, by the definition of transitive closure, $B=t(A)=\sum_{k=1}^{m} A^k$.

Example 2.1. Let $A = \begin{pmatrix} (1,0,0) & (0,0,0) \\ (0,0,0) & (0,1,1) \end{pmatrix}$.

Then, $A^2 = \begin{pmatrix} (1,0,0) & (0,0,0) \\ (0,0,0) & (0,1,1) \end{pmatrix}$.

Then, $t(A) = A + A^2$

$= \begin{pmatrix} (1,0,0) & (0,0,0) \\ (0,0,0) & (0,1,1) \end{pmatrix} + \begin{pmatrix} (1,0,0) & (0,0,0) \\ (0,0,0) & (0,1,1) \end{pmatrix} = \begin{pmatrix} (1,0,0) & (0,0,0) \\ (0,0,0) & (0,1,1) \end{pmatrix}$.

Theorem 2.2. Let $A\in\mu_m$ be a mFM. If either $A^q \leq A^r$ or $A^r \leq A^q$ holds for every $q < r$, then A converges.

Proof: Let $A=(a_{ij})$ and $A^q\leq A^r$ for every $q < r$, then $a_{ij}{}^{(q)} \leq a_{ij}{}^{(r)}$

$\Longrightarrow a_{ijn}{}^{(q)} \leq a_{ijn}{}^{(r)}$ and $a_{ijp}{}^{(q)} \leq a_{ijp}{}^{(r)}$.

Therefore, $a_{ijn}{}^{(q)} \leq a_{ijn}{}^{(r)} \leq a_{ijn}{}^{(r+1)} \leq a_{ijn}{}^{(r+2)} \leq \ldots\ldots.$

And

$a_{ijn}{}^{(q)} \leq a_{ijn}{}^{(r)} \leq a_{ijn}{}^{(r+1)} \leq a_{ijn}{}^{(r+2)} \leq\ldots\ldots\leq a_{ijp}{}^{(t)} = a_{ijp}{}^{(t+1)}=\ldots$

for finite naturals numbers s, t, simply, a finite numbers of distinct mFM occurs in the power of A, hence, A converges.

Similarly, it can be shown that, A converges when $A^r \leq A^q$ for every q < r.

Definition 2.5. Let $A=(a_{ij}) \in \mu_m$ be a mFM. A is said to be row diagonally dominant if $a_{ij} \leq a_{ii}$ $(1 \leq i, j \leq m)$, A matrix A is called diagonally dominant if it is both row and column diagonally dominant.
Diagonally dominant property is very important in the matrix and its determinant theory, here we mention some results using this.

Theorem 2.3. Let $A \in \mu_m$ be a mFM. If $A^q \leq A^r$ for every q < r and A is row column diagonally dominant, then A converges to A^l for some $l \leq m-1$.
Proof: Let A be row diagonally dominant. Now,
$a_{ij_n}{}^{(k)} = \sum_{j_2, j_3, \ldots, j_{k-1}} a_{ij_1 n} a_{j_1 j_2 n} a_{j_2 j_3 n} \ldots\ldots a_{j_{k-1} i_n}$
$\leq \sum_{j_1} a_{ij_1 n} = a_{ijn}$
Similarly, $a_{iip}^{(k)} \leq a_{iip}$, therefore, $a_{ii}^{(k)} \leq a_{ii}$. On the other hand, $a_{ii} \leq a_{ii}^{(k)}$ $(k \geq 1)$, also, using the lemma we conclude that, $A^{m-1} = A^m$.
Hence, A converges to A^l for some $l \leq m-1$.

Theorem 2.4. Let $A \in \mu_m$ be mFM. If $A^q \leq A^r$ for every q < r and A is row or column diagonally dominant, then A is power convergent and converged to t(A).
Proof: From the previous said theorem, if $A^q \leq A^r$ for every q < r then A converges,
Taking q = 1, r = 2, we get $A \leq A^2$. Similarly, $A^2 \leq A^3 \leq A^4 \leq \ldots\ldots$
Now, $t(A) = \sum_{K=1}^{m} A^k = A + A^2 + A^3 + \ldots\ldots + A^m$
Again, since A is row or column diagonally dominant, A converges to A^l for some $l \leq m-1$.
Then,
$A \leq A^2 \leq A^3 \leq \ldots\ldots \leq A^l = A^{l+1} = A^{l+2} = \ldots\ldots = A^m$
Therefore $t(A) = A^l$.
Then A is the power converges to
$t(A) = A + A^2 + A^3 = A + A^2 + A^2 = A^2$ (since $A^3 = A$).

3. Permanent and its properties

The permanent has a rich structure when restricted to certain classes of matrices, particularly, matrices of zeros and once, (entry wise) non negative matrices and (+ve) semidefinite matrices. furthermore, there is a certain similarly of its properties over the classes of non-negative matrices and the class of (+ve) semi definite matrices.

In this part, permanent of m-polar fuzzy matrices defined with some examples. Some properties due to permanent nature of mFM is also discussed here. Also, a method to evaluate the permanent for large order mFM is described.

Definition 3.1. (Permanent) If $A=[aij]_{mxn}$ is a crisp matrix of order $n \times n$, then the permanent of A is denoted by per(A) and defined as $\text{per}(A) = \sum_{\sigma \in S_N} \prod_{i=}^{n} a_{i\sigma(i)}$.
where, S_n the symmetric group of order n.

Here is an example to illustrate the permanent of a crisp matrix.
Example 3.1.

Let $A = \begin{pmatrix} 7 & 2 & 1 \\ 7 & 7 & 2 \\ 7 & 7 & 3 \end{pmatrix}$ be a crisp matrix of order 3×3.

Then, per(A) = $a_{11}.a_{22}.a_{33}+a_{12}.a_{23}.a_{31}+a_{13}.a_{22}.a_{31}+ a_{11}.a_{23}.a_{32}+a_{12}.a_{21}.a_{33}+a_{13}.a_{21}.a_{32}$

= 7.7.3+2.2.3+1.7.7+7.2.7+2.7.3+1.7.7

= 419

Definition 3.2. Let $\tilde{A}=[a_{ij}]_{pxq}$ be a m-polar fuzzy matrix, where, $a_{ij}=(a_{ij}^{(1)},a_{ij}^{(2)}, \ldots\ldots,a_{ij}^{(m)})$, And $0\leq a_{ij}^{(k)}\leq 1$, for all k.

Then the permanent of $\tilde{A}$ is denoted by per($\tilde{A}$) and defined by

per ($\tilde{A}$)=$\sum_{\sigma \in s} \prod_{i=1}^{p} a_{i\sigma(i)}$ for $p \leq q$

[where S is the set of all one to one mapping from {1,2,…..,p} to{1,2,….,q}]

And

per ($\widetilde{A}$)=$\sum_{\sigma \in s} \prod_{i=1}^{q} a_{i\sigma(i)}$ for $p > q$

[where S is the set of all one to one mapping from {1,2,…..,q} to{1,2,….,p}]

Two expression are written for the permanent of matrix, because for p > q, there are one-to-one mapping from {1,2,…..,q} to{1,2,….,p}.in this case no one-to-one mappings are possible from {1,2,…..,p} to{1,2,….,q}.

Following example are consider to illustrate the definition.

Example 3.2.

(i) Let

$$\tilde{A} = \begin{pmatrix} (0.3,0.4,0.6) & (0.8,0.7,0.1) & (0.5,0.6,0.9) \\ (0.4,0,7,0.5) & (0.9,0.6,0.2) & (0.1,0.7,0.5) \end{pmatrix}$$

Therefore, per ($\widetilde{A}$)=max{ min ((0.3,0.4,0.6),(0.9,0.6,0.2)),
min((0.3,0.4,0.6),(0.1,0.7,0.6)), min ((0.8,0.7,0.1),(0.4,0.7,0.5)),
min ((0.8,0.7,0.1),(0.1,0.7,0.5)),min((0.5,0.6,0.9),(0.4,0.7,0.5)),
min((0.5,0.6,0.9),(0.9,0.6,0.2))}
= max {(0.3,0.4,0.2),(0.1,0.4,0.5),(0.4,0.7,0.1), (0.1,0.7,0.1), (0.4,0.6,0.5),(0.5,0.6,0.2)}
= (0.5,0.7,0.5).

(ii) $\tilde{B}=\begin{pmatrix} (0.2,0.4,0.6) & (0.3,0.6,0.5) \\ (0.6,0.8,0.1) & (0.7,0.9,0.2) \\ (0.4,0.3,0.5) & (0.6,0.3,0.3) \end{pmatrix}$

per($\widetilde{B}$)= max { min((0.2,0.4,0.6),(0.7,0.9,0.2)), min((0.2,0.4,0.6),(0.6,0.3,0.3))
min((0.6,0.8,0.1),(0.3,0.6,0.5)), min((0.6,0.8,0.1),(0.6,0.3,0.3))
min((0.4,0.3,0.5),(0.3,0.6,0.5)), min((0.4,0.3,0.5),(0.7,0.9,0.2))}
= max{(0.2,0.4,0.2),(0.2,0.3,0.3),(0.3,0.6,0.1),
(0.6,0.3,0.1),(0.3,0.3,0.5),(0.4,.3,0.2)}
= (0.6,0.6,0.5)

(iii)

Let $\tilde{C}=\begin{pmatrix} (0.1,0.3,0.6) & (0.3,0.5,0.1) & (0.2,0.4,0.3) \\ (0.5,0.3,0.7) & (0.6,0.6,0.4) & (0.2,0.3,0.4) \\ (0.1,0.3,0.5) & (0.3,0.1,0.7) & (0.1,0.3,0.7) \end{pmatrix}$

per($\widetilde{C}$)=max{ min((0.1,0.3,0.6),(0.6,0.6,0.4),(0.1,0.3,0.7))
min((0.3,0.5,0.1),(0.2,0.3,0.5),(0.1,0.3,0.5))
min((0.2,0.4,0.3),(0.5,0.3,0.7),(0.3,0.1,0.7))
min((0.2,0.4,0.3),(0.6,0.6,0.4),(0.1,0.3,0.5))
min((0.3,0.5,0.1),(0.5,0.3,0.7),(0.1,0.3,0.7))
min((0.1,0.3,0.6),(0.2,0.3,0.5),(0.3,0.1,0.7))}
= (0.1,0.3,0.5).

3.1. Some properties of mFMs

Some trivial properties of permanent of mFMs are presented below.

1. For any triangular or diagonal mFM $\tilde{A}$,
per($\widetilde{A}$)=min{of its diagonal entries}.

2. For any row MFM or column mFM $\tilde{A}$
per($\widetilde{A}$)=max{of the entries}.

3. If $\tilde{A}$ and $\tilde{B}$ are any two mFM, such that both $\widetilde{AB}$ and $\widetilde{BA}$ are defined, then
per(AB)≠per(BA).

In general the proof of the above properties are straight forward, they are illustrated of the following examples :

Example 3.3.

Let $A=\begin{pmatrix} (0.1,0.2,0.3) & (0.4,0.6,0.5) & (0.5,0.3,0.6) \\ (0,0,0) & (0.1,0.5,0.6) & (0.4,0,0.6) \\ (0,0,0) & (0,0,0) & (0.2,0.,3,0) \end{pmatrix}$

Then

per($\tilde{A}$)=(0.1,0.2,0.3)(0.1,0.5,0.6)(0.2,0.3,0)+(0.1,0.2,0.3)(0.4,0,0.6)(0,0,0)+ (0.4,0.6,0.5)(.4,0,0.6)(0,0,0)+(0.4,0.6,0.5)(0,0,0)(0.2,0.3,0)+(0.5,0.3,0.1)(0,0,0)(0,0,0)+ (0.5,0.3,0.1)(0.1,0.5,0.6)(0,0,0)

per($\widetilde{A}$) = (0.1,0.2,0)+(0,0,0)+(0,0,0)+(0,0,0)+(0,0,0)+(0,0,0)
= (0.1,0.2,0).

Example 3.4.

Let $A=\begin{pmatrix} (0.6,0.2,0.3) & (0.2,0.4,0.1) & (0.5,0.3,0.1) \\ (0.8,0.5,0.9) & (0.4,0.3,0.6) & (0.2,0.4,0.6) \\ (0.4,0.7,0.5) & (0.3,0.7,0.6) & (0.5,0.7,0.2) \end{pmatrix}$ and

$B=\begin{pmatrix} (0.1,0.3,0.6) & (0.3,0.5,0.1) & (0.2,0.4,0.3) \\ (0.5,0.3,0.7) & (0.6,0.6,0.4) & (0.2,0.3,0.5) \\ (0.1,0.3,0.5) & (0.3,0.1,0.7) & (0.1,0.3,0.7) \end{pmatrix}$

Then

$$AB=\begin{pmatrix}(0.2,0.3,0.7) & (0.3,0.4,0.4) & (0.2,0.3,0.5)\\(0.4,0.3,0.6) & (0.4,0.5,0.6) & (0.2,0.4,0.6)\\(0.3,0.3,0.6) & (0.3,0.6,0.4) & (0.2,0.4,0.5)\end{pmatrix}$$

and $BA=\begin{pmatrix}(0.3,0.5,0.3) & (0.3,0.4,0.6) & (0.2,0.4,0.2)\\(0.6,0.5,0.5) & (0.4,0.3,0.7) & (0.5,0.4,0.4)\\(0.3,0.3,0.7) & (0.3,0.3,0.6) & (0.2,0.3,0.6)\end{pmatrix}=$

We have, per(AB)=(0.2,0.3,0.4), per(BA) =(0.2,0.3,0.2).
Therefore, per(AB) $\neq$ per(BA), in general.

Proposition 3.1. For any mFM $\tilde{A}$, per($\tilde{A}$)=per($\widetilde{A^T}$).
Proof: Let $\tilde{A}=[\widetilde{a_{ij}}]_{nxp}$ where $\widetilde{a_{ij}}=(a_{ij}^1,a_{ij}^2,\dots\dots,a_{ij}^m)$,
When n $\leq$ p
$\text{per}(\widetilde{A})=\sum_{\sigma\epsilon s}\prod_{i=1}^{n}\widetilde{a_{i\sigma(i)}}=\sum_{\sigma\epsilon s}\prod_{i=1}^{n}(a_{i\sigma(i)}^1,a_{i\sigma(i)}^2,\dots\dots,a_{i\sigma(i)}^m)$
Let $\widetilde{A^T}=\tilde{B}$ = $[b_{ij}]_{pxn}$, p $\geq$ n.
Then $b_{ij}=a_{ji}$
i.e, $(b_{ij}^1,b_{ij}^2,\dots\dots,b_{ij}^m)=(a_{ji}^1,a_{ji}^2,\dots\dots,a_{ji}^m)$
also
per($\widetilde{A^T}$)=per ($\widetilde{B}$)=$\sum_{\sigma\epsilon s}\prod_{j=1}^{n}b_{\sigma(j)j}=\sum_{\sigma\epsilon s}\prod_{j=1}^{n}a_{j\sigma(j)}$
$=\sum_{\sigma\epsilon s}\prod_{i=1}^{n}a_{i\sigma(i)}=\text{per}(\widetilde{A})$.
For m > n, the proof is similar as before.

Proposition 3.2. Interchanging of rows or columns does not effect to the permanent value of the matrix.
Proof: Let $(\widetilde{A})=[a_{ij}]_{nxp}$ be an mFM of order $n\times p$ and $\widetilde{B}=(b_{ij})_{nxp}$ is obtained from $\tilde{A}$ by interchanging the *r*th and *s*th row (r < s) of $\widetilde{A}$. Then it is clear that,
$B_{ij}=a_{ij}$, i $\neq$ r, j $\neq$ s and $b_{rj}=a_{sj}$, $b_{sj}=a_{rj}$
i.e. $(b_{ij}^1,b_{ij}^2,\dots\dots,b_{ij}^m)=(a_{ij}^1,a_{ij}^2,\dots\dots,a_{ij}^m)$, i $\neq$ r, j $\neq$ s
and, $(b_{rj}^1,b_{rj}^2,\dots\dots,b_{rj}^m)=(a_{sj}^1,a_{sj}^2,\dots\dots,a_{sj}^m)$, $(b_{sj}^1,b_{sj}^2,\dots\dots,b_{sj}^m)=(a_{rj}^1,a_{rj}^2,\dots\dots,a_{rj}^m)$.

Now, per($\widetilde{B}$) = $\sum_{\sigma\epsilon s}(\prod_{i=1}^{n}\widetilde{b_{i\sigma(i)}})$
$=\sum_{\sigma\epsilon s}b_{1\sigma(1)}\,b_{2\sigma(2)}\dots\dots b_{r\sigma(r)}\dots\dots b_{s\sigma(s)}\dots\dots b_{n\sigma(n)}$
$=\sum_{\sigma\epsilon s}a_{1\sigma(1)}\,a_{2\sigma(2)}\dots\dots a_{s\sigma(r)}\dots\dots b_{r\sigma(s)}\dots\dots b_{n\sigma(n)}$
$=\sum_{\sigma\epsilon s}\{(a_{1\sigma(1)}^1,a_{1\sigma(1)}^2,\dots\dots,a_{1\sigma(1)}^m)(a_{2\sigma(2)}^1,a_{2\sigma(2)}^2\dots\dots,a_{2\sigma(2)}^m)\dots$
$(a_{s\sigma(r)}^1,a_{s\sigma(r)}^2\dots\dots,a_{s\sigma(r)}^m)$
$\dots(a_{r\sigma(s)}^1,a_{r\sigma(s)}^2\dots\dots,a_{r\sigma(s)}^m)\dots(a_{n\sigma(n)}^1,a_{n\sigma(n)}^2\dots\dots,a_{n\sigma(n)}^m)\}$
Let $\lambda=\begin{pmatrix}1\ 2\cdots r\cdots s\cdots\cdots n\\1\ 2\cdots s\cdots r\cdots\cdots n\end{pmatrix}$

and $\sigma\lambda=\phi$ then
$\phi(i)=\sigma(i)$ for i ≠ r, i ≠ s and $\phi(r)=\sigma(s)$ and $\phi(s)=\sigma(r)$.

Then, per$(\widetilde{B})$ =
$\sum_{\phi\epsilon s}\{(a^1_{1\phi(1)}, a^2_{1\phi(1)}, \ldots\ldots, a^m_{1\phi(1)})(a^1_{2\phi(2)}, a^2_{2\phi(2)} \ldots\ldots, a^m_{2\phi(2)})\ldots$
$(a^1_{s\phi(s)}, a^2_{s\phi(s)} \ldots\ldots, a^m_{s\phi(s)})$
...$(a^1_{r\phi(r)}, a^2_{r\phi(r)} \ldots\ldots, a^m_{r\phi(r)})\ldots(a^1_{n\phi(n)}, a^2_{n\phi(n)} \ldots\ldots, a^m_{n\phi(n)})\}$
i.e. per$(\widetilde{B})=\sum_{\phi\epsilon s}\{(a^1_{1\phi(1)}, a^2_{1\phi(1)}, \ldots\ldots, a^m_{1\phi(1)})$
$(a^1_{2\phi(2)}, a^2_{2\phi(2)} \ldots\ldots, a^m_{2\phi(2)})\ldots(a^1_{r\phi(r)}, a^2_{r\phi(r)} \ldots\ldots, a^m_{r\phi(r)})$
...$(a^1_{s\phi(s)}, a^2_{s\phi(s)} \ldots\ldots, a^m_{s\phi(s)})\ldots(a^1_{n\phi(n)}, a^2_{n\phi(n)} \ldots\ldots, a^m_{n\phi(n)})$
= per $(\widetilde{A})$.
Hence, interchanging of row or column does not alter the permanent value.

Example 3.5.

Let $\check{A} = \begin{pmatrix} (0.2,0.4,0.6) & (0.1,0.4,0.3) & (0.2,0.5,0.9) \\ (0.6,0.5,0.3) & (0.4,0.2,0.6) & (0.2,0.4,0.6) \\ (0.4,0.7,0.8) & (0.1,0.5,0.7) & (0.3,0.2,0.5) \end{pmatrix}$

Then, per$(\widetilde{A})$ = (0.2,0.5,0.6).

Now, $(\tilde{A})^T = \begin{pmatrix} (0.2,0.4,0.6) & (0.6,0.5,0.3) & (0.4,0.7,0.8) \\ (0.1,0.4,0.3) & (0.4,0.2,0.6) & (0.1,0.5,0.7) \\ (0.2,0.5,0.9) & (0.2,0.4,0.6) & (0.3,0.2,0.5) \end{pmatrix}$.

Then, per$(\widetilde{A})$ = (0.2,0.5,0.6).
Therefore, per$(\widetilde{A})$ = per$(\widetilde{A^T})$.

4. g-inverse and regularity of m-polar fuzzy matrix

If $\tilde{A}$ and $\widetilde{B}$ be two m-polar fuzzy matrices satisfying the relation $\tilde{A}\tilde{B}\tilde{A}=\tilde{A}$, then $\widetilde{B}$ is called g-inverse of $\tilde{A}$ and $\widetilde{A}$ is called regular.

Proposition 4.1. If $\widetilde{A} = [a_{ij}]_{p\times q}$ be a mFM, where $a_{ij}=(a^1_{ij}, a^2_{ij}, \ldots\ldots, a^m_{ij})$, and $\tilde{B}$ is g-inverse of $\widetilde{A}$, then per$(\tilde{A}\tilde{B})$ = per $(\widetilde{AB})^2$**.**
Proof: Since $\tilde{B}$ is a g-inverse of $\tilde{A}$
then $\tilde{A}\tilde{B}\tilde{A}=\tilde{A}$
⇒ $\tilde{A}\tilde{B}\tilde{A}\tilde{B}=\tilde{A}\tilde{B}$
⇒ $(\tilde{A}\tilde{B})^2=\tilde{A}\tilde{B}$
Then per $(\tilde{A}\tilde{B})$ = per$(\tilde{A}\tilde{B})^2$.
Proposition 4.1. Let $\tilde{A}=[a_{ij}]_{pxq}$ be a mFM, where $a_{ij}=(a^1_{ij}, a^2_{ij}, \ldots\ldots, a^m_{ij})$, and $0 \le a^k_{ij} \le 1$ for all k and, if a row be multiplied by a scalar k, then the permanent value will be k.per$(\tilde{A})$.
Proof: Let, $\tilde{A}=[a_{ij}]_{pxq}$ be a mFM of order $p \times q$ and $\tilde{B}=[b_{ij}]_{pxq}$ be another mFM obtained by multiplying k to a row of $\tilde{A}$.
Let $a_{ij} = (a^1_{ij}, a^2_{ij}, \ldots\ldots, a^m_{ij})$, $b_{ij}=(b^1_{ij}, b^2_{ij}, \ldots\ldots, b^m_{ij})$ and k is multiplied to the *r*th row.

Let p ≤ q, then

per($\tilde{B}$)=$\sum_{\sigma\epsilon s}\prod_{i=1}^{p} b_{i\sigma(i)}$

=$\sum_{\sigma\epsilon s}\{(a^{1}_{1\sigma(1)}, a^{2}_{1\sigma(1)}, \dots\dots, a^{m}_{1\sigma(1)})\quad (a^{1}_{2\sigma(2)}, a^{2}_{2\sigma(2)} \dots\dots, a^{m}_{2\sigma(2)})$

$(a^{1}_{3\sigma(3)}, a^{2}_{3\sigma(3)} \dots\dots, a^{m}_{3\sigma(3)})$

...k$(a^{1}_{r\sigma(r)}, a^{2}_{r\sigma(r)} \dots\dots, a^{m}_{r\sigma(r)})\dots(a^{1}_{p\sigma(p)}, a^{2}_{p\sigma(p)} \dots\dots, a^{m}_{p\sigma(p)})\}$

=k$\sum_{\sigma\epsilon s}\prod_{i=1}^{p} a_{i\sigma(i)}$

=k per($\widetilde{A}$).

Proof is similar for m > n.

5. Application of m-polar fuzzy matrix

m-polar fuzzy concept, set and matrices not only have applications in mathematical theories but also it is applied in real world problems, such as

(i) It is useful to explore weighted games cooperatives and multivalued relations.

(ii) In decision making issues, e.g. when country elects its political leaders, a company decided to manufacture an item or product, in communication issues, over a noisy channel, a communication channel may have a different types of network range, radio frequency, band width and latency. In social network, the influence rate of different people may have different w.r.t socialism, pro-activeness and trading relationship.

(iii) In case of the operation on permanent of matrix, finding the permanent of square matrix is equivalent to finding :

   (a) number of perfect matching in the bipartite graph (bi-adjacency matrix)

   (b) number of cycles cover in the directed graph

Thus it plays an important role in various domains of science and technology.

6. Conclusion

The m-polar fuzzy concept is a very important and essential tool to model a large number of problems of both in mathematics and real life. Therefore, it has vast field of applications. In this paper, we first introduce m-polar fuzzy relation and m-polar fuzzy matrix based on m-polar fuzzy algebras. Also, some result on transitive closure and power of convergence are investigated. Some well known mathematical operations with permanent are applied on m-polar fuzzy matrix and its behaviors and observed. Many beautiful and useful works are done by many researchers and will be done in future. Here we conclude this paper with the hope for doing more better works on this topic.

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