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Strong Vertex Covering in Hypergraphs

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Abstract. In this paper we consider strong vertex covering sets and strong vertex covering number of a hypergraph. We consider a single operation of removing a vertex from the hypergraph. We prove that the strong vertex covering number of the resulting sub hypergraph (or partial sub-hypergraph) does not exceed the strong vertex covering number of the hypergraph. We also consider an independent set in hypergraphs and we prove that the independence number decreases when a vertex is removed from the hypergraphs. Some examples have also been given.

Keywords: Vertex covering set, Minimum vertex covering set, Vertex covering number, Strong vertex covering set, Strong vertex covering number, Stable set, Maximum Stable set, Stability number, Independent set, independence number

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1. Introduction

The concepts of vertex covering and stability in graphs can be further defined in hypergraphs also. In fact these concepts for hypergraphs can be found in [3]. For hypergraphs, however there are related concepts which have been defined. These concepts are called strong vertex covering and independence. All these four concepts have key roles in the theory of coloring of hypergraphs [3].

In this paper we consider strong vertex covering and independence for different purpose. We study the effect of removing a vertex (from the hypergraph) on the strong vertex covering number and the independence number of the hypergraphs.

2. Notations and preliminaries

For the definition of hypergraph and other terminology one can refer [3]**.** However we make some conventions.

(1) No two edges in a hypergraph will intersect in more than one vertex.

(2) The empty set will not be regarded as an edge but an edge may have only one vertex.

If G is a hypergraph then V (G) or V will denote the set of all vertices and E (G) or E will denote the set of all edges of G.

Definition 2.1: Stable set [3]

A set $S \subseteq V(G)$ in a hypergraph G is a Stable set if no edge is a subset of S. A Stable set with maximum cardinality is called a maximum stable set of G and is denoted as β_0-set of G.

The cardinality of a maximum Stable set is called the Stability number of G and is denoted as $\beta_0(G)$.

Definition 2.2. Vertex covering set [7]
A subset S of V (G) is called a vertex covering set if every edge of G has non-empty intersection with S.
A vertex covering set with minimum cardinality is called α_0-set of G.
The cardinality of a minimum vertex covering set of G is called the vertex covering number of G and is denoted as α_0.

Definition 2.3. Strong vertex covering set [7]
Let G be a hypergraph and $S \subseteq V(G)$ then S is said to be strong vertex covering set if whenever x and y are adjacent in G then $x \in S$ or $y \in S$.

Definition 2.4. Minimum strong vertex covering set [7]
A strong vertex covering set with minimum cardinality is called minimum strong vertex covering set and it is denoted as $\alpha_s -$set of G.

Definition 2.5. Strong vertex covering number [7]
The cardinality of a minimum strong vertex covering set ($\alpha_s -$set) is called the strong vertex covering number of the hypergraph G and it is denoted as $\alpha_s(G)$.

Definition 2.6. Independent set [3]
Let G be a hypergraph and $S \subseteq V(G)$ then S is called an independent set of G if whenever x and y belongs to S and $x \neq y$, they are non-adjacent vertices of G.

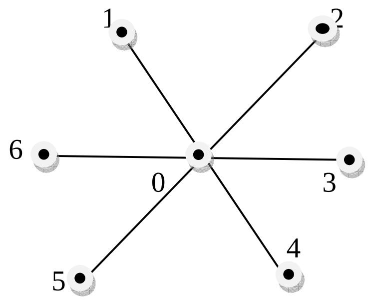
Definition 2.7. Independence number [3]
The cardinality of a maximum independent set of a hypergraph G is called the independence number of G and it is denoted as $\beta_s(G)$

It is obvious that,
(1) Every independent set is a stable set. However a stable set need be an independent set.
(2) A set S is independent if and only if V(G) \ S is a strong vertex covering set of G.
Also note that
(i) A set T is an independent if and only if every edge intersects T in at most one point and
(ii) A set S is a strong vertex covering set if and only if for every edge e there is a vertex v in e such that e \ {v} is a subset of S.(equivalently for every e in V(G) \S and for every edge e containing v, e \ {v} is a subset of S)

Remarks 2.8. (1) A set S is a maximum independent set of G if and only if V(G) \ S is a minimum strong vertex covering set of G.Hence $\alpha_s(G) + \beta_s(G) = n =$the number of vertices in hypergraph G.

(2) If G has no isolated vertex then obviously every strong vertex covering set is an H-dominating set of G.
(3) Every strong vertex covering set is a vertex covering set but a vertex covering set need not be a strong vertex covering set.

Example 2.9. Let G=(V,E) be a hypergraph whose vertex set V(G)={01,2,3,4,5,6} and edge set E(G)={ {1,0,4},{2,0,5},{3,0,6} }.



In this hypergraph the set {0}is a vertex covering set but it is not a strong vertex covering set. Here T= {1, 2, 3, 4, 5, 6} is a Stable set but it is not an independent set because 1 and 4 are adjacent vertices.
Let G be a hypergraph, $v \in V(G)$. Here we consider the sub-hypergraph G \{v} Whose vertex set is V(G) \{v} and the edge set is equal to e'=e \ {v} for some edge e of G. First we prove that Strong vertex covering number does not increase when a vertex is removed from a hypergraph.

3. Main results

Theorem 3.1. Let G be a hypergraph and $v \in V(G)$ then $\alpha_s(G \setminus \{v\}) \le \alpha_s(G)$.

Proof: Let S be a minimum strong vertex covering set of G.
Case I. Suppose $v \notin S$.
Let x and y be two vertices of G \ {v} which are adjacent in G \{v}. Then they are adjacent in G also. Then x and y are adjacent in G.
Since S is a strong vertex covering set $x \in S$ or $y \in S$.Therefore S is a strong vertex covering set of G \ {v}. Thus $\alpha_S(G \setminus \{v\}) \le \alpha_S(G)$.
Case II. Suppose $v \in S$.
Consider the set S_1=S \ {v}. Let x and y be vertices of G \ {v} which are adjacent in G \ {v}. Then as proved above $x \in S$ or $y \in S$.
Since $x \ne v$ and $y \ne v$, $x \in S_1$ or $y \in S_1$.Thus S_1 is a strong vertex covering set of G \ {v}. So $\alpha_S(G \setminus \{v\}) \le |S_1| < |S| = \alpha_S(G)$. Thus $\alpha_S(G \setminus \{v\}) \le \alpha_S(G)$.
Hence from both cases $\alpha_s(G \setminus \{v\}) \le \alpha_s(G)$. ■

Next we prove the necessary and sufficient condition under which the strong vertex covering number is decreases.

Theorem 3.2. $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$ if and only if there is a minimum strong vertex covering set S such that $v \in S$.

Proof: Suppose S is a minimum strong vertex covering set of G such that $v \in S$.
Let S_1=S \ {v}. Then as proved in theorem 3.1 S_1 is a strong vertex covering set of G \ {v}. Thus $\alpha_s(G \setminus \{v\}) \le |S_1| < |S| = \alpha_s(G)$. Hence $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$.

Conversely suppose $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$. Let S_1 is a minimum vertex covering set of G \ {v}. Then S_1 cannot be a strong vertex covering set of G. (Becauseotherwise $\alpha_s(G) \le |S_1| = \alpha_s(G \setminus \{v\})$).

Let $S = S_1 \cup \{v\}$ then S is a strong vertex covering set of G.

Since $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$, S is a strong vertex covering set of G and it also contain a vertex v. ■

Corollary 3.3. $\alpha_s(G \setminus \{v\}) = \alpha_s(G)$ if and only if v does not belongs to any minimum strong vertex covering set of G. ■

We consider the following two notations here.

$$V_s^- = \{v \in V(G) : \alpha_s(G \setminus \{v\}) < \alpha_s(G)\}$$

$$V_s^0 = \{v \in V(G) : \alpha_s(G \setminus \{v\}) = \alpha_s(G)\}$$

Remark 3.4. From the above theorem 3.2 and its corollary 3.3 it follows that

1. $V_s^- = \cup\{S : S$ is a minimum strong vertex covering set of G}
2. Similarly $V_s^0 = \cup\{V(G) \setminus S : S$ is a minimum strong vertex covering set of G}
3. Since union of strong vertex covering set is a strong vertex covering set, V_s^- is a strong vertex covering set.
4. Since intersection of independent sets is independent set, V_s^0 is a independent set.

Obviously complement of a strong vertex covering set is an independent set, complement of a minimum strong vertex covering set is a maximum independent set.

Hence we have $\alpha_s(G) + \beta_s(G) = n =$ the number of vertices in hypergraph G.

Consider the above facts, the following theorems have been stated without proof.

Theorem 3.5. If G is a hypergraph and $v \in V(G)$ then $\beta_s(G \setminus \{v\}) \le \beta_s(G)$. ■

Theorem 3.6. If G is a hypergraph and $v \in V(G)$ then $\beta_s(G \setminus \{v\}) < \beta_s(G)$ if and only if v belongs to every maximum independent subset of G. ■

Theorem 3.7. If G is a hypergraph and $v \in V(G)$ then $\beta_s(G \setminus \{v\}) = \beta_s(G)$ if and only if $v \notin S$ for some maximumindependent subset S of G. ■

Remark 3.8. Since V_s^0 is an independent set, if $v \in V(G)$

1. Then $N(v) \subseteq T$ for every strong vertex covering set T of G. Thus we have $N(v) \subseteq \cap\{T : T$ is a strong vertex covering set of G}

Thus if T is a strong vertex covering set of G then $N(v) \subseteq T$ which implies that $|T| \ge \deg v \ge \delta(G)$. Hence $\alpha_s(G) \ge \delta(G)$.

(where $\delta(G) = \min\{\deg v : v \in V(G)\}$)

2. Again $\alpha_s(G) + \beta_s(G) = n =$ The number of vertices in hypergraph G.

Therefore $\alpha_s(G \setminus \{v\}) + \beta_s(G \setminus \{v\}) = n - 1.$

Also it has been proved $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$.Thus when the strong vertex covering number decreases (when a vertex is removed from the hypergraphs) the independence number remains same and also when strong vertex covering number remains same the independence number decreases.

Consider the hypergraph G and $v \in V(G)$. Here we will consider the partial sub-hypergraph G \ {v} whose vertex set is V (G) \ {v} and having edges have those edges of G which do not contain v.

Now we prove that the strong vertex covering number does not increase when a vertex is removed from the hypergraphs.

Theorem 3.9. Let G be a hypergraph and $v \in V(G)$ then $\alpha_s(G \setminus \{v\}) \le \alpha_s(G)$.

Proof: Let S be a minimum strong vertex covering set of G.

Case I. $v \notin S$.

If x and y are adjacent vertices of G \ {v} then x and y are adjacent vertices of G.
Since S is a strong vertex covering set of G, $x \in S$ or $y \in S$.

Thus S is a strong vertex covering set of G \ {v}.

Therefore $\alpha_s(G \setminus \{v\}) \le |S| = \alpha_s(G).$

Case II. Suppose $v \in S$.

Let $S_1 = S \setminus \{v\}$ then S_1 is a subset of V (G \ {v}).

Let x and y are adjacent vertices of G \ {v} then x and y are adjacent vertices of G. Since S is a strong vertex covering set of G, $x \in S$ or $y \in S$.

Since $v \notin \{x, y\}, x \in S_1$ or $y \in S_1$. Therefore S_1 is a strong vertex covering set of G \ {v}.

Thus $\alpha_s(G \setminus \{v\}) \le |S_1| < |S| = \alpha_s(G).$

Therefore $\alpha_s(G \setminus \{v\}) \le \alpha_s(G)$.

Hence from both cases $\alpha_s(G \setminus \{v\}) \le \alpha_s(G)$.■

Theorem 3.10. Let G be a hypergraph and $v \in V(G)$.If there is a $\alpha_s -$set S such that $v \in S$ then $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$.

Proof: Consider the set $S_1 = S \setminus \{v\}$.

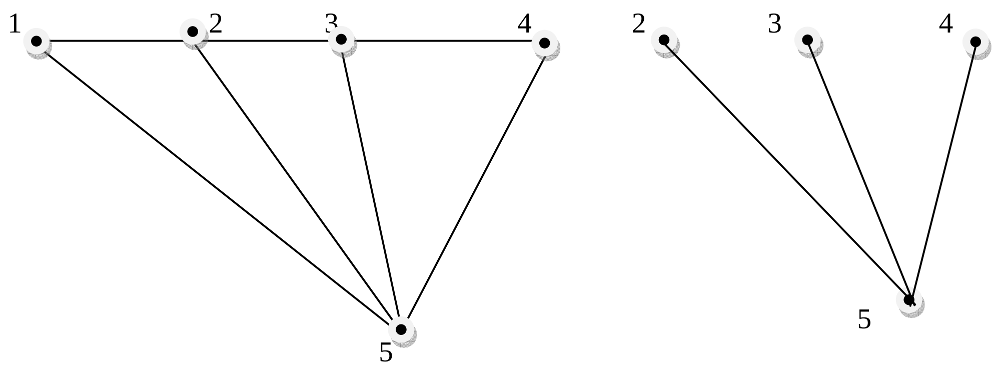
We prove that $S_1$is a strong vertex covering set of G \ {v}.

For this we suppose that x and y are vertices of G \ {v} which are adjacent in G \ {v}. So there is an edge e of G \ {v} such that $x, y \in e$. Since e is also an edge of G, it follows that x and y is adjacent in G. Since S is a strong vertex covering set of G, $x \in S$ or $y \in S$ .Since $x \ne v$ and $y \ne v, x \in S_1$ or $y \in S_1$. Thus S_1 is a strong vertex covering set of G \ {v}. Thus $\alpha_S(G \setminus \{v\}) \le |S_1| < |S| = \alpha_S(G)$.

Hence $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$. Thus the theorem is proved. ■

Remark 3.11. The above theorem 3.10 says that if $v \in S$ where S is a minimum strong vertex covering set of G then $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$. However the above condition is not necessary for $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$.

Example 3.12. Consider the hypergraph G whose vertex set is V(G)={1,2,3,4,5} and edge set is E(G)={ {1,2,3,4},{2,5},{3,5},{4,5}}.



Note that the set S= {2, 3, 4} is a minimum strong vertex covering set of G. Hence $\alpha_s(G) = 3.$

Now consider the hypergraph G \ {1}. In this hypergraph the edges are {2, 5}, {3, 5} and {4, 5}. In this hypergraph {5} is a minimum strong vertex covering set of G \ {1}.Hence $\alpha_s(G \setminus \{1\}) = 1.$ Thus $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$.

However note that $1 \notin S.$ also note that there is no α_s –set of G which contains 1.

Now we introduce the following notations for hypergraph G.

$V_{cr}^{-} = \{v \in V(G) : \alpha_s(G \setminus \{v\}) < \alpha_s(G)\}$

$V_{cr}^{0} = \{v \in V(G) : \alpha_s(G \setminus \{v\}) = \alpha_s(G)\}$

Accordingly if S is a minimum strong vertex covering set of G then for every vertex v in S, $v \in V_{cr}^{-}$.(From theorem 3.10). Thus if S_1, S_2… S_k is all minimum strong vertex covering set of G then $\cup S_i (i = 1, 2, ..., k) \subseteq V_{cr}^{-}$.

Now we prove a necessary and sufficient condition under which a vertex $v \in V_{cr}^{0}$.

Theorem 3.13. Let G be a hypergraph and $v \in V(G)$ then $\alpha_s(G \setminus \{v\}) = \alpha_s(G)$ if and only if there is a minimum strong vertex covering set S_1 of G \ {v} such that $N(v) \subseteq S_1$

Proof: First suppose that $\alpha_s(G \setminus \{v\}) = \alpha_s(G)$.

Let S be a α_s –set of G. If $v \in S$ then $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$ this is a contradiction. Thus $v \notin S$ and therefore S is a strong vertex covering set of G \ {v}. Hence S is a α_s – set of G \ {v}.

Let S_1=S.

Suppose $N(v) \not\subset S_1$ then there is a neighbor w of v such that $w \notin S_1$. Then $v \notin S$ and $w \notin S$.Which is a contradicts the fact that S is a strong vertex covering set of G. Thus $N(v) \subseteq S_1$.

Conversely suppose S_1 is a $\alpha_s -$set of G \ {v} such that $N(v) \subseteq S_1$.

We claim that S_1 is a strong vertex covering set of G.

To prove this suppose x and y are adjacent vertices in G.

If $x \neq v$ and $y \neq v$, x and y are adjacent in G\ {v} then $x \in S_1$ or $y \in S_1$.

If $x \neq v$ and $y \neq v$, suppose x and y adjacent in G but not in G \ {v} then every edge e which contains x and y also contains v.

Therefore x and y are neighbors of v and hence $x, y \in S_1$.

Suppose $x = v$ and $y \neq v$ then y is a neighbor of v, because x and y are adjacent in G. Hence $y \in S_1$ because $N(v) \subseteq S_1$. Thus $\alpha_s(G) \leq |S| \leq \alpha_s(G \setminus \{v\}) \leq \alpha_s(G)$.

Hence $\alpha_s(G \setminus \{v\}) = \alpha_s(G)$.

This completes the proof of the theorem. ■

From first part of the above theorem 3.13 it is clear that if $\alpha_s(G \setminus \{v\}) = \alpha_s(G)$ then $N(v) \subseteq S_1$ for every $\alpha_s -$set of G. Hence we have the following corollary.

Corollary 3.14. Let G is a hypergraph and $v \in V(G)$ if $\alpha_s(G \setminus \{v\}) = \alpha_s(G)$ then $N(v) \subseteq \cap\{S : S$ is a $\alpha_s -$set of G}.■

Corollary 3.15. Let G be a hypergraph then V_{cr}^0 is an independent subset of G.

Proof: Suppose u and $v \in V_{cr}^0$ with $u \neq v$. If u and v are adjacent then $u \in N(v)$ and hence by corollary 3.14 $u \in S$ for every $\alpha_s -$set of G.

Let S_0 be any $\alpha_s -$set of G then $u \in S_0$. Hence by theorem 3.13 $u \in V_{cr}^-$, which is a contradiction as $v \in V_{cr}^0$.

Therefore u and v cannot be adjacent. Thus V_{cr}^0 is an independent subset of G. ■

Now we consider maximum independent sets in hypergraphs.

Theorem 3.16. Let G be a hypergraph and $v \in V(G)$.Then $\beta_s(G \setminus \{v\}) < \beta_s(G)$ if and only if there is a maximum independent set T of G \ {v} such that $N(v) \cap T = \emptyset$.

Proof: Suppose $\beta_s(G \setminus \{v\}) < \beta_s(G)$.

Let T be a maximum independent set of G. If $v \in T$ then T \ {v} is a maximum independent set of G \ {v}. Since v is not adjacent to any vertex in G, $N(v) \cap T = \emptyset$.

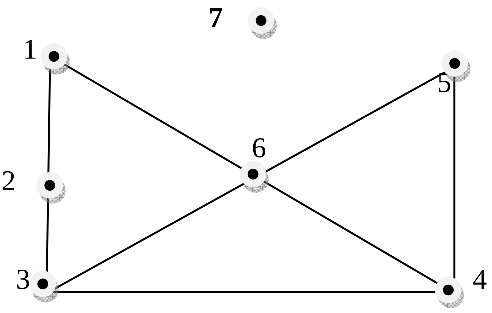
Suppose $v \notin T$ for any maximum independent set T of G. Then for any set T, T is an independent set in G \ {v}, which implies that $\beta_s(G \setminus \{v\}) \geq \beta_s(G)$. This is a contradiction.

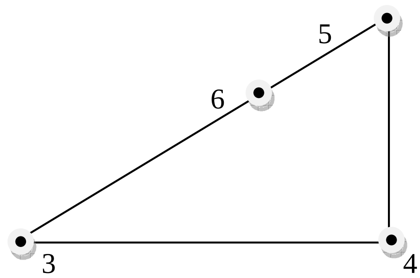
Thus it is impossible that $v \notin T$ for every maximum independent set T of G.
Conversely suppose T is a maximum independent set of G \ {v} such that $N(v) \cap T = \emptyset$.
We claim that T is an independent set in G also.
Suppose $x, y \in T$ which are adjacent in G. Then $x \neq v$ and $y \neq v$. As $N(v) \cap T = \emptyset$, x and y are adjacent in G \ {v} which contradicts the independent set of T in G \ {v}.
Let $T_1 = T \cup \{v\}$. Since $N(v) \cap T = \emptyset$, T_1 is an independent set in G and hence $\beta_s(G) \geq |T_1| > |T| = \beta_s(G \setminus \{v\})$. Thus $\beta_s(G \setminus \{v\}) < \beta_s(G)$. ■

Corollary 3.17. Let G be a hypergraph and $v \in V(G)$ if $\beta_s(G \setminus \{v\}) < \beta_s(G)$ then there is a maximum independent set S of G such that $v \in S$. ■

The converse of above corollary 3.17 is not true in general.

Example 3.18. Consider the hypergraph G whose vertex set V(G)={1,2,3,…7} and edges are {1,2,3},{1,6,4},{3,4},{3,6},{4,5} and {5,6}. Note that 7 is an isolated vertex in G. We can observe that S= {1, 5, 7} is a maximum independent set of G and $\beta_s(G) = 3$.





Now consider the sub-hypergraph (G \ {1}). The edges of this hypergraph are {3,4},{4,5},{5,6} and {3,6}. In this sub-hypergraph T= {3, 5, 7} is a maximum independent set and hence $\beta_s(G \setminus \{1\}) = 3$.

Thus $\beta_s(G \setminus \{v\}) = \beta_s(G)$ although $1 \in S$. This is a maximum independent set of G.

We may note that $\alpha_s(G \setminus \{v\}) = \alpha_s(G) - 1$ is not always true if $\alpha_s(G \setminus \{v\}) < \alpha_s(G)$.

Example 3.19. Consider hypergraph as given in 2.9.Here the set S= {0, 1, 2, 3} is a α_s −set of G. Hence $\alpha_s(G) = 4$.
In the sub-hypergraph (G \{0}) there are no edges and its vertex set is{1,2,3,4,5,6}.
Thus $\alpha_s(G \setminus \{0\}) = 0$. Hence $\alpha_s(G \setminus \{0\}) = \alpha_s(G) - 4$.

Theorem 3.20. Let G be a hypergraph and $v \in V(G)$ then $\alpha_s(G \setminus \{v\}) < \alpha_s(G) - k$ if and only if $\beta_s(G \setminus \{v\}) = \beta_s(G) + k - 1$ for every integer $k \geq 0$ and $k \leq \alpha_s(G)$.
Proof: Let n be the number of the vertices of a hypergraph G.
Here $\alpha_s(G \setminus \{v\}) + \beta_s(G \setminus \{v\}) = n - 1$.

Suppose $\alpha_s(G\setminus\{v\})=\alpha_s(G)-k$. Then $\alpha_s(G)-k+\beta_s(G\setminus\{v\})=n-1$.

Therefore $\beta_S(G\setminus\{v\})=(n-\alpha_S(G))+k-1$.

Hence $\beta_s(G\setminus\{v\})=\beta_s(G)+k-1$.

The converse can be proved in a similar manner. ■

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