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An Analytical Technique for Solving Nonlinear over Damped Vibrating Systems which vary with Time

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Abstract. Based on the general Struble's technique and extended KBM method a simple analytical solution has been developed to determine approximate solutions for nonlinear over damped vibrating systems with varying coefficients. The determination of the solution is more systematic and easier than the existing procedures developed by several authors. The method is illustrated by an example.

Keywords: Nonlinear system, over-damped vibrating system, varying coefficient, perturbed equation, eigenvalues

AMS Mathematics Subject Classification (2010): 34E05

1. Introduction

The asymptotic method Krylov-Bogoliubov-Mitroplshkii (KBM) [1-3] is particularly convenient and extensively used methods to study nonlinear differential systems with small nonlinearities. Originally, the method was developed by Krylov and Bogoliubov [1] for obtaining periodic solution of a second order nonlinear differential equation. Letter, the method was amplified and justified mathematically by Bogoliubov and Mitropolishkii [2,3]. Popov [4] extended the method to a damped oscillatory process in which a strong linear damping force acts. Murty, Dekshatulu and Krisna [5] extended the method to over-damped nonlinear system. Shamsul [6-8] investigated over-damped nonlinear systems and found approximate solutions of *Duffing*'s equation when one root of the unperturbed equation was respectively double or triples of the other. Shamsul [9] has presented a unified method for solving an *n*-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes with slowly varying coefficient. Pinakee *et.al* [10] has presented extended KBM method for under-damped, damped and over-damped vibrating systems in which the coefficients change slowly and periodically with time. Pinakee *et.al* [11] extended the technique for damped forced nonlinear system with varying coefficients. Recently Shamsul [12] has developed the general Struble's techniques for several damping effect. The aim of this paper is to find a solution of over damped nonlinear vibrating systems that vary slowly with time in which one of the eigen-values is multiple (almost two hundred times; *i.e.*, Bicentuple) of the other eigen-value and measure better result for strong nonlinearities.

2. Method

Let us consider a nonlinear differential system governed by

$$\ddot{x} + 2k_1(\tau)\dot{x} + (k_2^2 + k_3\cos\tau)x = -\varepsilon\, f(x, \dot{x}, \tau), \qquad \tau = \varepsilon t \tag{1}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, k_1, k_2 and k_3 are constants, $k_2 = O(\varepsilon) = k_3$, $\tau = \varepsilon t$ is the slowly varying time, $k_1(\tau) \geq 0$, f is a given nonlinear function. We set $\omega^2(\tau) = (k_2^2 + k_3\cos\tau)$, where $\omega(\tau)$ is known as internal frequency.

Putting $\varepsilon = 0$ and $\tau = \tau_0$ = constant, in Eq.(1), we obtain the unperturbed solution of (1) in the form

$$x(t,0) = x_{1,0}e^{\lambda_1(\tau_0)t} + x_{-1,0}e^{\lambda_2(\tau_0)t}, \tag{2}$$

Let Eq.(1) has two eigenvalues, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ are constants, but when $\varepsilon \neq 0$, $\lambda_1(\tau_0)$ and $\lambda_2(\tau_0)$ vary slowly with time. We may consider that $|\lambda_2(\tau_0)| >> |\lambda_1(\tau_0)|$. When $\varepsilon \neq 0$, we seek a solution of Eq. (1) in the form

$$x(t,\varepsilon) = x_1(t,\tau) + x_{-1}(t,\tau) + \varepsilon u_1(x_1, x_{-1}, t, \tau) + \varepsilon^2 u_2(x_1, x_{-1}, t, \tau) + ..., \tag{3}$$

where x_1 and x_{-1} satisfy the equations

$$\begin{aligned} \dot{x}_1 &= \lambda_1(\tau)x_1 + \varepsilon X_1(x_1, x_{-1}, \tau) + \varepsilon^2 X_1(x_1, x_{-1}, \tau)..., \\ \dot{x}_{-1} &= \lambda_2(\tau)x_{-1} + \varepsilon X_{-1}(x_1, x_{-1}, \tau) + \varepsilon^2 X_{-1}(x_1, x_{-1}, \tau)..., \end{aligned} \tag{4}$$

Differentiating $x(t,\varepsilon)$ two times with respect to t, substituting for the derivatives $\ddot{x}$ and x in the original equation (1) and equating the coefficient of ε, we obtain

$$\begin{aligned} &(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1})X_1 + (\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1})X_{-1} + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1} + \\ &+ (\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_1)(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_2)u_1 \\ &= -f^{(0)}(x_1, x_{-1}, \tau), \end{aligned} \tag{5}$$

where $\lambda_1' = \dfrac{d\lambda_1}{d\tau}$, $\lambda_2' = \dfrac{d\lambda_2}{d\tau}$, $Dx_1 = \dfrac{\partial}{\partial x_1}$, $Dx_{-1} = \dfrac{\partial}{\partial x_{-1}}$, $f^{(0)} = f(x_0, \dot{x}_0, \tau)$

Herein it is assumed that $f^{(0)}$ can be expanded in Taylor's series

$$f^{(0)} = \sum_{r_1, r_2 = 0}^{\infty} F_{r_1, r_2}(\tau)x_1^{r_1}x_{-1}^{r_2} \tag{6}$$

To obtain a special over-damped solution of (1), we impose a restriction that $u_1 \cdots$ exclude the terms $x_1^{i_1}x_{-1}^{i_2}, i_1\lambda_1 + i_2\lambda_2 < (i_1 + i_2)k(\tau_0)$, $i_1, i_2 = 0, 1, 2\cdots$. The assumption assures that $u_1 \cdots$ are free from secular type terms $te^{-\lambda_1 t}$. This restriction guarantees that the solution always excludes *secular*-type terms or the first harmonic terms, otherwise a sizeable error would occur [12]. Moreover, we assume that X_1 and X_{-1} respectively contains terms x_1 and x_{-1}.

3. Example

As example of the above procedure, let us consider a nonlinear vibrating system with slowly varying coefficients

$$\ddot{x} + 2k_1(\tau)\dot{x} + (k_2^2 + k_3 \cos\tau)x = -\varepsilon x^3 \tag{7}$$

Here over dots denote differentiation with respect to t. $x_0 = x_1 + x_{-1}$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(x_1^3 + 3x_1^2 x_{-1} + 3x_1 x_{-1}^2 + x_{-1}^3). \tag{8}$$

We substitute in (5) and separate it into two parts as

$$\begin{aligned}&\left(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}\right) X_1 + \left(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}\right) X_{-1} + \lambda_1' x_1 + \lambda_2' x_{-1} - \lambda_2 X_1 - \lambda_1 X_{-1}\\ &= -(x_1^3 + 3x_1^2 x_{-1} + x_{-1}^3)\end{aligned} \tag{9}$$

And $\left(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_1\right)\left(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1} - \lambda_2\right) u_1 = -(3x_1 x_{-1}^2)$ (10)

The particular solution of (10) is

$$u_1 = c_1\, x_1 x_{-1}^2 \tag{11}$$

where $c_1 = -3/2\lambda_1(\lambda_1 + \lambda_2)$

Now we have to solve (9) for two functions X_1 and X_{-1} (discussed in section 2)

The particular solutions are

$$\left(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}\right) X_1 + \lambda_1' x_1 - \lambda_2 X_1 = -(x_1^3 + 3x_1^2 x_{-1}) \tag{12}$$

and $\left(\lambda_1 x_1 Dx_1 + \lambda_2 x_{-1} Dx_{-1}\right) X_{-1} + \lambda_2' x_{-1} - \lambda_1 X_{-1} = -x_{-1}^3$ (13)

The particular solution of (12)-(13) is

$$X_1 = \lambda_1' x_1 n_1 + n_2 x_1^3 + n_3 x_1^2 x_{-1}, \text{ and } X_{-1} = \lambda_2' x_{-1} l_1 + l_2 x_{-1}^3, \tag{14}$$

where

$n_1 = -1/(\lambda_1 - \lambda_2)$, $n_2 = -1/(3\lambda_1 - \lambda_2)$, $n_3 = -3/2\lambda_1, l_1 = 1/(\lambda_1 - \lambda_2)$, $l_2 = -1/(3\lambda_2 - \lambda_1)$

Substituting the functional values of X_1, X_{-1} into (4) and rearranging, we obtain

$$\dot{x}_1 = \lambda_1 x_1 + \varepsilon\left(\lambda_1' x_1 n_1 + n_2 x_1^3 + n_3 x_1^2 x_{-1}\right) \tag{15}$$

and

$$\dot{x}_{-1} = \lambda_2 x_{-1} + \varepsilon\left(\lambda_2' x_{-1} l_1 + l_2 x_{-1}^3\right) \tag{16}$$

Therefore, the first order solution of (7) is

$$x(t,\varepsilon) = x_1 + x_{-1} + \varepsilon u_1, \tag{17}$$

where x_1, x_{-1} are given by (15), (16) and u_1 is given by (11).

4. Results and discussions

A technique is presented based on the general Struble's technique and extended KBM method to determine approximate solutions of over damped nonlinear vibrating systems which vary with time. The solution has been determined under the technique which gives better result for long time. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented Struble's technique and extended KBM method of this article, we refer to the works of Murty, Dekshatulu and Krishna [5] Shamsul [6-9] and Pinakee *et al* [10-11]. In this paper we have compared the perturbation solution (17) to those obtained by Runge-Kutta (Fourth order) method.

First of all, x is calculated by (18) with initial conditions $[x(0)=1.0\,, \dot{x}=0.0]$ or $x_1=1.0$, $x_{-1}=-.170814$ for $\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}$, $\lambda_1=-.04$, $\lambda_2=-7$, $\varepsilon=.9$. Then corresponding numerical solutions is also computed by Runge-Kutta fourth-order method. The result is shown in Fig.1. We see that in Fig. 1 the perturbation solution nicely agree with the numerical solution. The corresponding numerical solutions have also been computed by Runge-Kutta fourth-order method. From Fig. 2, Fig. 3, Fig. 4, Fig. 5 and Fig. 6, we observe that the approximate solutions agree with numerical results nicely and give desired result for strong nonlinearity.

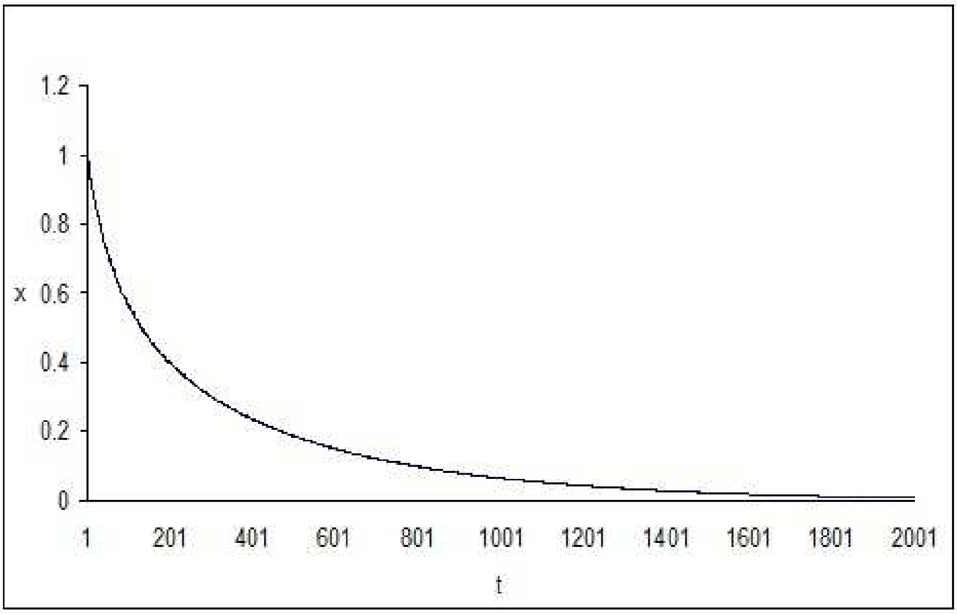

Figure 1: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $[x(0)=1.0\,, \dot{x}=0.0]$ or $x_1=1.0$, $x_{-1}=-.170814$ for $\omega=\omega_0\sqrt{(k_2^2+k_3\cos\tau)}$, $\lambda_1=-.04$, $\lambda_2=-7$, $\varepsilon=.9$,

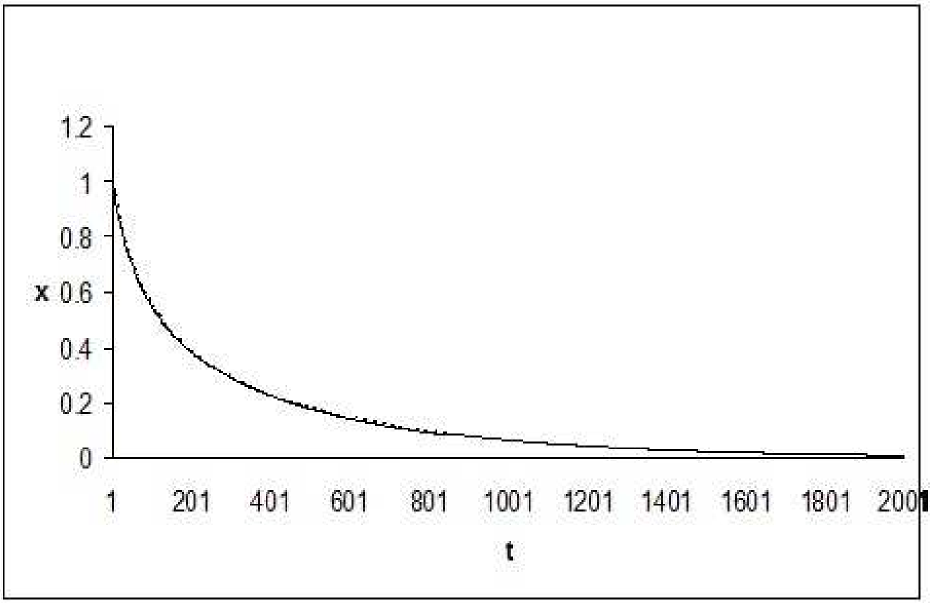

Figure 2: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $[x(0) = 1.0\,, \dot{x} = 0.0\,]$

or $x_1 = 1.0\,,\ x_{-1} = -.185349$ for $\omega = \omega_0\sqrt{(k_2^2 + k_3\cos\tau)}\,,\ \lambda_1 = -.04,\ \ \lambda_2 = -7\,,$ $\varepsilon = 1$

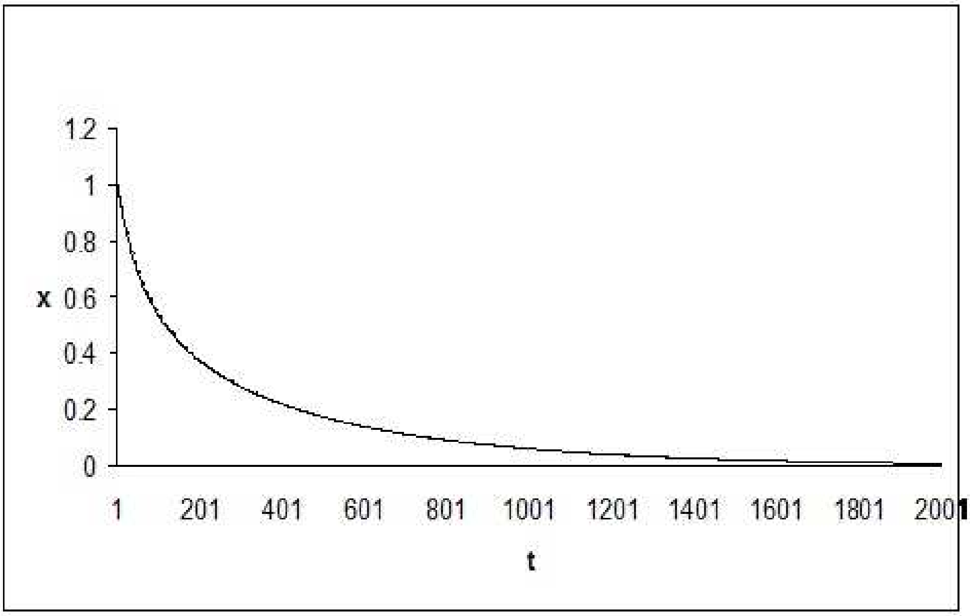

Figure 3: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $[x(0) = 1.0\,, \dot{x} = 0.0\,]$

or $x_1 = 1.0$, $x_{-1} = -.199889$ for $\omega = \omega_0 \sqrt{(k_2^2 + k_3 \cos\tau)}$, $\lambda_1 = -.04$, $\lambda_2 = -7$, $\varepsilon = 1.1$

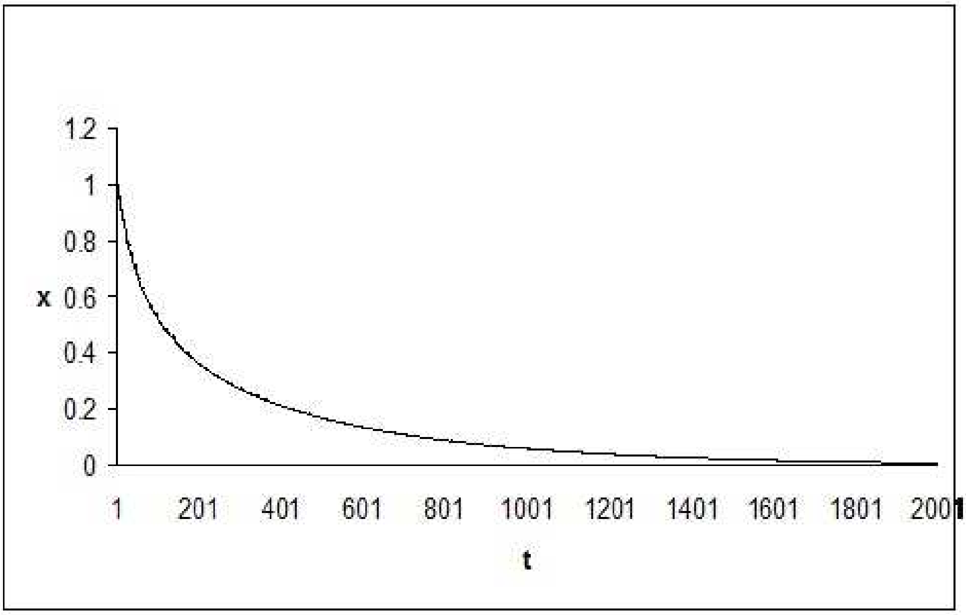

Figure 4: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $[x(0) = 1.0, \dot{x} = 0.0]$

or $x_1 = 1.0$, $x_{-1} = -.214419$ for $\omega = \omega_0 \sqrt{(k_2^2 + k_3 \cos\tau)}$, $\lambda_1 = -.04$, $\lambda_2 = -7$, $\varepsilon = 1.2$

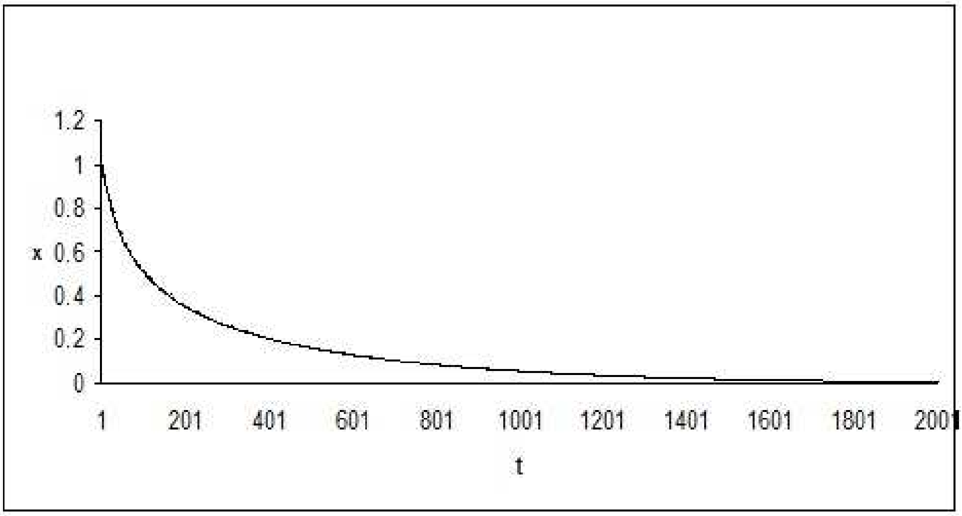

Figure 5: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $[x(0) = 1.0, \dot{x} = 0.0]$

or $x_1 = 1.0$, $x_{-1} = -.228952$ for $\omega = \omega_0 \sqrt{(k_2^2 + k_3 \cos\tau)}$, $\lambda_1 = -.04$, $\lambda_2 = -7$, $\varepsilon = 1.3$

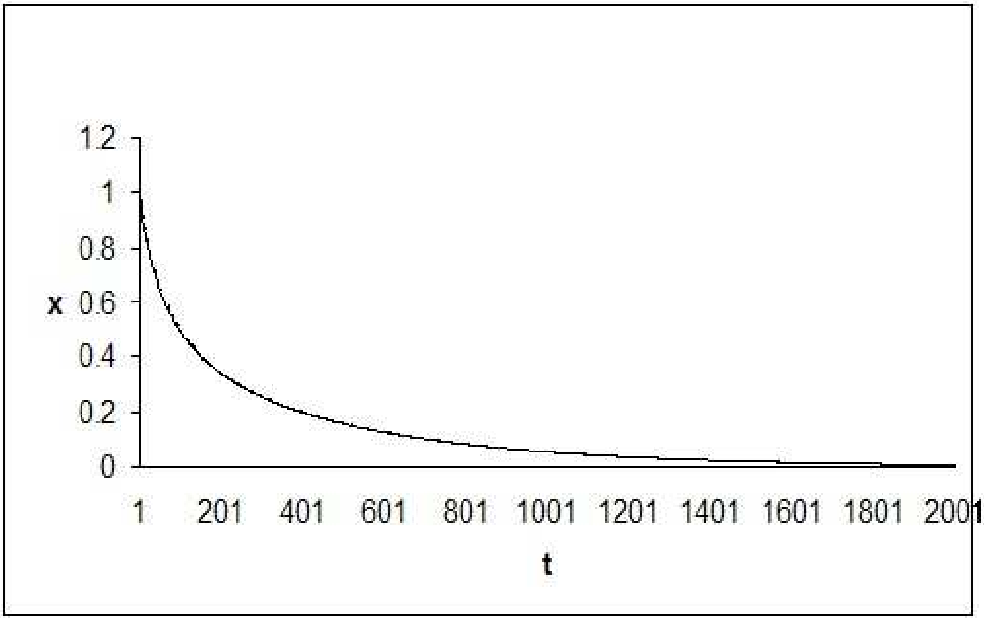

Figure 6: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $[x(0) = 1.0\,, \dot{x} = 0.0]$

or $x_1 = 1.0$, $x_{-1} = -.243488$ for $\omega = \omega_0 \sqrt{(k_2^2 + k_3 \cos\tau)}$, $\lambda_1 = -.04$, $\lambda_2 = -7$, $\varepsilon = 1.4$.

5. Conclusion

In this article a technique is developed for obtaining the solution of over-damped nonlinear vibrating system where the coefficients change slowly with time and when one of the eigen-values is multiple (almost two hundred times; *i.e.*, Bicentuple) of the other eigen-value. The solution is simpler than classical KBM method. It gives better result and agrees nicely with the numerical solutions for strong nonlinearity. The method can be preceded to higher order nonlinear systems.

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