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Certain New Classes Containing Combination of Ruscheweyh Derivative and a New Generalized Multiplier Differential Operator

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Abstract. Certain new classes containing the linear operator obtained as a linear combination of Ruscheweyh derivative and a new generalized multiplier differential operator have been considered. Sharp results concerning coefficients, distortion theorems of functions belonging to these classes are discussed.

Keywords: Analytic function, differential operator, multiplier transformation

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1. Introduction

Denote by U the open unit disc of the complex plane, $U = \{z \in C : |z| < 1\}$. Let $H(U)$ be the space of holomorphic functions in U. Let A denote the family of functions in $H(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

The author has recently introduced the following new generalized multiplier differential operator in [16].

Definition 1.1. Let $m \in N_0 = N \cup \{0\}, \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then for $f \in A$, a new generalized multiplier operator $I^m_{\alpha,\beta}$ was defined by

$$I^0_{\alpha,\beta} f(z) = f(z), I^1_{\alpha,\beta} f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha + \beta}, \ldots, I^m_{\alpha,\beta} f(z) = I_{\alpha,\beta}(I^{m-1}_{\alpha,\beta} f(z)).$$

Remark 1.2. Observe that for $f(z)$ given by (1.1), we have

$$I^m_{\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} A_k(\alpha, \beta, m) a_k z^k, \tag{1.2}$$

where

$$A_k(\alpha,\beta,m)=\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m. \tag{1.3}$$

We note that: i) $I^m_{1-\beta,\beta,0}f(z)=D^m_\beta f(z),\ \beta\ge 0$ (See F. M. Al-Oboudi [1]), ii) $I^m_{l+1-\beta,\beta,0}f(z)=I^m_{l,\beta}f(z),\quad l>-1,\quad \beta\ge 0$ (See A. Catas [3] and he has considered for $l\ge 0$) and iii) $I^m_{\alpha,1}f(z)=I^m_\alpha f(z),\alpha>-1$(See Cho and Srivastava [4]) and Cho and Kim [5]). $D^m_1 f(z)$ was introduced by Salagean [9] and was considered for $m\ge 0$ in [2].

Definition 1.3. ([8]) For $m\in N_0$, $f\in A$, the operator R^m is defined by $R^m: A\to A$,

$R^0 f(z)=f(z)$, $R^1 f(z)=zf'(z),\ldots,$

$(m+1)R^{m+1}f(z)=z(R^m f(z))'+mR^m f(z), z\in U.$

Remark 1.4. If $f(z)=z+\sum_{k=2}^\infty a_k z^k\in A$, then $R^m f(z)=z+\sum_{k=2}^\infty B_k(m)a_k z^k$, $z\in U$, where

$$B_k(m)=\frac{(m+k-1)!}{m!(k-1)!}. \tag{1.4}$$

The author has introduced the following operator in [17].

Definition 1.5. Let $f\in A, m\in N_0=N\cup\{0\}, \delta\ge 0, \beta\ge 0, \alpha$ a real number such that $\alpha+\beta>0$. Denote by $RI^m_{\alpha,\beta,\delta}$, the operator given by $RI^m_{\alpha,\beta,\delta}: A\to A,$ $RI^m_{\alpha,\beta,\delta}f(z)=(1-\delta)R^m f(z)+\delta I^m_{\alpha,\beta}f(z), z\in U.$

The operator was studied also in [11], [12], [13], [14] and [15]. Clearly $RI^m_{\alpha,\beta,,0}=R^m$ and $RI^m_{\alpha,\beta,,1}=I^m_{\alpha,\beta}$.

Remark 1.6. If $f(z)=z+\sum_{k=2}^\infty a_k z^k$, then from (1.2) and Remark 1.4, we have

$$RI^m_{\alpha,\beta,\delta}f(z)=z+\sum_{k=2}^\infty\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k,\quad z\in U,$$

where $A_k(\alpha,\beta,m)$ and $B_k(m)$ are as defined in (1.3) and (1.4), respectively.

We introduce new classes as below by making use of the generalized operator $RI^m_{\alpha,\beta,\delta}$.

Definition 1.7. Let $f\in A, m\in N_0=N\cup\{0\}, \delta\ge 0, \rho\in[0,1), \sigma\in(0,1], \beta\ge 0, \alpha$ a real number such that $\alpha+\beta>0$. Then $f(z)$ is in the class $S^m_{\alpha,\beta,\delta}(\sigma,\rho)$ if and only if

$$\left|\frac{\frac{z(RI^m_{\alpha,\beta,\delta}f(z))'}{RI^m_{\alpha,\beta,\delta}f(z)}-1}{\frac{z(RI^m_{\alpha,\beta,\delta}f(z))'}{RI^m_{\alpha,\beta,\delta}f(z)}+1-2\rho}\right|<\sigma, z\in U. \tag{1.5}$$

Definition 1.8. Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1), \sigma \in (0,1], \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $K^m_{\alpha,\beta,\delta}(\sigma, \rho)$ if and only if

$$\left| \frac{\frac{[z^2 (RI^m_{\alpha,\beta,\delta} f(z))']'}{(RI^m_{\alpha,\beta,\delta} f(z))'} - 1}{\frac{z(RI^m_{\alpha,\beta,\delta} f(z))'}{RI^m_{\alpha,\beta,\delta} f(z)} + 1 - 2\rho} \right| < \sigma \quad , z \in U . \tag{1.6}$$

Definition 1.9. Let $f \in A, m \in N_0 = N \cup \{0\}, \delta \geq 0, \rho \in [0,1), \sigma \in (0,1], \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$.Then $f(z)$ is in the class $C^m_{\alpha,\beta,\delta}(\sigma, \rho)$ if and only if

$$\left| \frac{\frac{[z(RI^m_{\alpha,\beta,\delta} f(z))']'}{(RI^m_{\alpha,\beta,\delta} f(z))} - 1}{\frac{[z(RI^m_{\alpha,\beta,\delta} f(z))']'}{(RI^m_{\alpha,\beta,\delta} f(z))'} + 1 - 2\rho} \right| < \sigma \quad , z \in U . \tag{1.7}$$

Definition 1.10. Let $f \in A, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \delta \geq 0, \rho \in [0,1), \sigma \in (0,1], \beta \geq 0,$ α a real number such that $\alpha + \beta > 0$.Then $f(z)$ is in the class $\mathrm{P}^m_{\alpha,\beta,\lambda,\delta}(\sigma, \rho)$ if and only if

$$\left| \frac{(1-\lambda)\frac{RI^m_{\alpha,\beta,\delta} f(z)}{z} + \lambda (RI^m_{\alpha,\beta,\delta} f(z))' + 1}{(1-\lambda)\frac{RI^m_{\alpha,\beta,\delta} f(z)}{z} + \lambda (RI^m_{\alpha,\beta,\delta} f(z))' + 1 - 2\rho} \right| < \sigma \, , z \in U . \tag{1.8}$$

Definition 1.11. Let $f \in A, m \in N_0 = N \cup \{0\}, \lambda \geq 0, \delta \geq 0, \rho \in [0,1), \sigma \in (0,1], \beta \geq 0,$ α a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\mathrm{H}^m_{\alpha,\beta,\lambda,\delta}(\sigma, \rho)$ if and only if

$$\left| \frac{(RI^m_{\alpha,\beta,\delta} f(z))' + \lambda z (RI^m_{\alpha,\beta,\delta} f(z))'' + 1}{(RI^m_{\alpha,\beta,\delta} f(z))' + \lambda z (RI^m_{\alpha,\beta,\delta} f(z))'' + 1 - 2\rho} \right| < \sigma \, , z \in U . \tag{1.9}$$

Let T denote the subclass of A consisting of functions whose non-zero coefficients, from second on, are negative; that is, an analytic function f is in T if and only if it can be expressed as $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0, z \in U$.If $f \in T$, then

$RI^m_{\alpha,\beta,\delta} f(z) = z - \sum_{k=2}^{\infty} \{(1-\delta) B_k(m) + \delta A_k(\alpha, \beta, m)\} a_k z^k$, where

$A_k(\alpha, \beta, m)$ and $B_k(m)$ are as defined in (1.3) and (1.4), respectively. We denote by $TS^m_{\alpha,\beta,\delta}(\sigma, \rho)$, $TK^m_{\alpha,\beta,\delta}(\sigma, \rho)$, $TC^m_{\alpha,\beta,,\delta}(\sigma, \rho)$, $T\mathrm{P}^m_{\alpha,\beta,\lambda,\delta}(\sigma, \rho)$ and $T\mathrm{H}^m_{\alpha,\beta,\lambda,\delta}(\sigma, \rho)$, the classes of functions $f(z) \in T$ satisfying (1.5), (1.6) (1.7),(1.8) and (1.9) respectively.

In this paper, sharp results concerning coefficients and distortion theorems for the classes $TS^m_{\alpha,\beta,\delta}(\sigma, \rho)$, $TK^m_{\alpha,\beta,\delta}(\sigma, \rho)$, $TC^m_{\alpha,\beta,,\delta}(\sigma, \rho)$, $T\mathrm{P}^m_{\alpha,\beta,\lambda,\delta}(\sigma, \rho)$ and

$TH^{m}_{\alpha,\beta,\lambda,\delta}(\sigma,\rho)$ are determined. Throughout this paper, unless otherwise mentioned we shall assume that $A_k(\alpha,\beta,m)$ and $B_k(m)$ are as defined in (1.3) and (1.4) respectively.

2. Coefficient bounds

In this section we study the characterization properties for functions in the classes $TS^{m}_{\alpha,\beta,\delta}(\sigma,\rho)$, $TK^{m}_{\alpha,\beta,\delta}(\sigma,\rho)$, $TC^{m}_{\alpha,\beta,,\delta}(\sigma,\rho)$, $TP^{m}_{\alpha,\beta,\lambda,\delta}(\sigma,\rho)$ and $TH^{m}_{\alpha,\beta,\lambda,\delta}(\sigma,\rho)$ are determined , following the papers of V. P. Gupta and P. K. Jain [6, 7] and H. Silverman[10].

Theorem 2.1. A function f is in $TS^{m}_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty}(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k \le 2\sigma(1-\rho). \qquad (2.1)$$

The result is sharp.

Proof. Suppose f satisfies (2.1).Then for $|z|<1$, we have

$$\begin{aligned}
&\left|z(RI^{m}_{\alpha,\beta,\delta}f(z))' - RI^{m}_{\alpha,\beta,\delta}f(z)\right| - \sigma\left|z(RI^{m}_{\alpha,\beta,\delta}f(z))' + (1-2\rho)RI^{m}_{\alpha,\beta,\delta}f(z)\right| = \\
&\quad \left|-\sum\nolimits_{k=2}^{\infty}(k-1)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k\right| - \\
&\quad \sigma\left|2(1-\rho)-\sum\nolimits_{k=2}^{\infty}(k+1-2\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k\right| \le \\
&\quad \sum\nolimits_{k=2}^{\infty}(k-1)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k - 2\sigma(1-\rho) + \\
&\quad \sum\nolimits_{k=2}^{\infty}\sigma(k+1-2\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k = \\
&\sum\nolimits_{k=2}^{\infty}\{(k-1+\sigma(k+1-2\rho)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k - 2\sigma(1-\rho) < 0.
\end{aligned}$$

Hence, by using the maximum modulus theorem and (1.5), $f \in TS^{m}_{\alpha,\beta,\gamma,\delta}(\sigma,\rho)$.

For the converse, assume that

$$\left|\frac{\frac{z(RI^{m}_{\alpha,\beta,\gamma,\delta}f(z))'}{RI^{m}_{\alpha,\beta,\gamma,\delta}f(z)}-1}{\frac{z(RI^{m}_{\alpha,\beta,\gamma,\delta}f(z))'}{RI^{m}_{\alpha,\beta,\gamma,\delta}f(z)}+1-2\rho}\right| = \left|\frac{-\sum_{k=2}^{\infty}(k-1)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}{2\sigma(1-\rho)-\sum_{k=2}^{\infty}\sigma(k+1-2\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}\right| < \sigma, z\in U.$$

Since $\mathrm{Re}(z)\le|z|$ for all $z\in U$, we obtain

$$\mathrm{Re}\left(\frac{\sum_{k=2}^{\infty}(k-1)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}{2\sigma(1-\rho)-\sum_{k=2}^{\infty}\sigma(k+1-2\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}\right) < \sigma. \quad (2.2)$$

Choose values of z on the real axis so that $\left(z(RI^{m}_{\alpha,\beta,\delta}f(z))' / RI^{m}_{\alpha,\beta,\delta}f(z)\right)$ is real. Upon clearing the denominator in (2.2) and letting $z\to 1$ through real values, we have the desired inequality (2.1).

The function

$f_1(z) = z - \frac{2\sigma(1-\rho)}{(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}} z^k, k \geq 2$ is an extremal function for the theorem.

Theorem 2.2. i) A function f is in $TK^m_{\alpha,\beta,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty} (k+1)(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}a_k \leq 4\sigma(1-\rho). \quad (2.3)$$

ii) A function f is in $TC^m_{\alpha,\beta,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty} k(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}a_k \leq 2\sigma(1-\rho). \quad (2.4)$$

The results (2.3) and (2.4) are sharp.

The proof of Theorem 2.2 is similar to that of Theorem 2.1 and so omitted. Extremal functions are given by

$$f_2(z) = z - \frac{4\sigma(1-\rho)}{(k+1)(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}} z^k, k \geq 2$$

and

$$f_3(z) = z - \frac{2\sigma(1-\rho)}{k(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}} z^k, k \geq 2,$$

respectively.

Theorem 2.3. (i) A function $f(z) \in T\mathrm{P}^m_{\alpha,\beta,\lambda,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty} (1+\lambda(k-1))(1+\sigma)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}a_k \leq 2\sigma(1-\rho). \quad (2.5)$$

ii) A function $f(z) \in T\mathrm{H}^m_{\alpha,\beta,\lambda,\delta}(\sigma,\rho)$ if and only if

$$\sum_{k=2}^{\infty} k(1+\lambda(k-1))(1+\sigma)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}a_k \leq 2\sigma(1-\rho). \quad (2.6)$$

The results (2.5) and (2.6) are sharp.

The proof of Theorem 2.3 is similar to that of Theorem 2.1 and so omitted. Extremal functions are given by

$$f_4(z) = z - \frac{2\sigma(1-\rho)}{(1+\lambda(k-1))(1+\sigma)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}} z^k, k \geq 2$$

and

$$f_5(z) = z - \frac{2\sigma(1-\rho)}{k(1+\lambda(k-1))(1+\sigma)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}} z^k, k \geq 2,$$

respectively.

Corollary 2.4.

i) If $f \in TS^m_{\alpha,\beta,\delta}(\sigma,\rho)$

then $a_k \le \dfrac{2\sigma(1-\rho)}{(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}, k \ge 2,$ with equality only for the functions of the form $f_1(z)$.

ii) If $f \in TK^m_{\alpha,\beta,\delta}(\sigma,\rho)$,

then $a_k \le \dfrac{4\sigma(1-\rho)}{(k+1)(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}, k \ge 2,$ with equality only for the functions of the form $f_2(z)$.

iii) If $f \in TC^m_{\alpha,\beta,\delta}(\sigma,\rho)$,

then $a_k \le \dfrac{2\sigma(1-\rho)}{k(k-1+\sigma(k+1-2\rho))\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}, k \ge 2,$ with equality only for the functions of the form $f_3(z)$.

iv) If $f(z) \in T\mathrm{P}^m_{\alpha,\beta,\lambda,\delta}(\sigma,\rho)$,

then $a_k \le \dfrac{2\sigma(1-\rho)}{(1+\lambda(k-1))(1+\rho)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}, k \ge 2,$ with equality only for the functions of the form $f_4(z)$.

v) If $f(z) \in T\mathrm{H}^m_{\alpha,\beta,\lambda,\delta}(\sigma,\rho)$,

then $a_k \le \dfrac{2\sigma(1-\rho)}{k(1+\lambda(k-1))(1+\sigma)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}, k \ge 2,$ with equality only for the functions of the form $f_5(z)$.

3. **Distortion theorems**

Theorem 3.1. If a function $f(z) \in T$ is in $TS^m_{\alpha,\beta,\delta}(\sigma,\rho)$ then

$$|f(z)| \ge |z| - \frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \le |z| + \frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U,$$

with equalities for

$$f(z) = z - \frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}z^2, (z \pm r).$$

Proof In view of Theorem 2.1, we have

$$(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}\sum_{k=2}^{\infty}a_k \leq$$

$$\sum_{k=2}^{\infty}(k-1+\sigma(k+1-2\rho)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k \leq 2\sigma(1-\rho).$$

Thus $\sum_{k=2}^{\infty}a_k \leq \frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,\delta)\}}$ .So we get for $z\in U$,

$$|f(z)| \leq |z|+|z|^2\sum_{k=2}^{\infty}a_k \leq |z|+\frac{\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2.$$

On the other hand

$$|f(z)| \geq |z|-|z|^2\sum_{k=2}^{\infty}a_k \geq |z|-\frac{2\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2.$$

Theorem 3.2. (i) If a function $f(z)\in T$ is in $T\ell^m_{\alpha,\beta,\delta}(\sigma,\rho)$ then

$$|f(z)| \geq |z|-\frac{4\sigma(1-\rho)}{3(1+\sigma(3-2\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z\in U$$

and

$$|f(z)| \leq |z|+\frac{4\sigma(1-\rho)}{3(1+\sigma(3-2\rho)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z\in U.$$

ii) If a function $f(z)\in T$ is in $T\Re^m_{\alpha,\beta,\delta}(\sigma,\rho)$ then

$$|f(z)| \geq |z|-\frac{\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z\in U$$

and

$$|f(z)| \leq |z|+\frac{\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z\in U.$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1.

Remark 3.3. The bounds of Theorem 3.2 are sharp since the equalities are attained for the functions $f(z)=z-\frac{4\sigma(1-\rho)}{3(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}z^2$ $(z=\pm r)$ and

$f(z)=z-\frac{\sigma(1-\rho)}{(1+\sigma(3-2\rho))\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}z^2$ $(z=\pm r)$, respectively

Theorem 3.4. (i) If a function $f(z)\in T$ is in $T\mathrm{P}^m_{\alpha,\beta,\delta}(\sigma,\rho)$ then

$$\left| f(z)\right| \geq |z| - \frac{2\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z\in U$$

and

$$\left| f(z)\right| \leq |z| + \frac{2\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z\in U .$$

ii) If a function $f(z)\in T$ is in $T\mathrm{H}^{m}_{\alpha,\beta,\delta}(\sigma,\rho)$ then

$$\left| f(z)\right| \geq |z| - \frac{\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z\in U$$

and

$$\left| f(z)\right| \leq |z| + \frac{\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z\in U .$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1.

Remark 3.5. The bounds of Theorem 3.2 are sharp since the equalities are attained for the functions $f(z)=z-\frac{2\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}z^2$ $(z=\pm r)$ and

$f(z)=z-\frac{\sigma(1-\rho)}{(1+\lambda)(1+\sigma)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}z^2$ $(z=\pm r)$, respectively

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