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# On Edge Neighborhood Graphs

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***Abstract.*** Let $G = (V, E)$ be a graph. The edge neighborhood graph $N_e(G)$ of $G$ is the graph with the vertex set $E \cup S$, where $S$ is the set of all open edge neighborhood sets of edges of $G$ and with two vertices $u, v$ in $N_e(G)$ adjacent if $u \in E$ and $v$ is an open edge neighborhood set containing $u$. In this paper, some properties of this new graph are established.

***Keywords:*** open edge neighborhood set, edge neighborhood graph

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## 1. Introduction

All graphs considered in this paper are finite, undirected without loops and multiple edges. We denote by $p$ the number of vertices and $q$ the number of edges of such a graph. Any undefined term here may be found in Kulli [1].

Let $G = (V, E)$ be a graph. For any edge $e \in E$, the open edge neighborhood $N(e)$ of $e$ is the set of edges adjacent to $e$. We call $N(e)$ is the open edge neighborhood set of an edge $e$ of $G$. Let $E = \{e_1, e_2, \ldots\ldots, e_q\}$ and let $S = \{N(e_1), N(e_2), \ldots, N(e_q)\}$ be the set of all open edge neighborhood sets of edges of $G$.

The neighborhood graph $N(G)$ of a graph $G = (V, E)$ is the graph with the vertex set $V \cup S$ where $S$ is the set of all open neighborhood sets of vertices of $G$ and with two vertices $u, v \in V \cup S$ adjacent if $u \in V$ and $v$ is an open neighborhood set containing $u$. This concept was introduced by Kulli in [2]. Many other graph valued functions in graph theory were studied, for example, in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]

The degree of an edge $uv$ is defined to be $\deg u + \deg v - 2$. An edge is called an isolated edge if $\deg uv = 0$.

The following will be useful in the proof of our result.

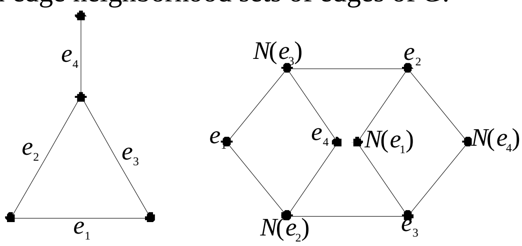
**Theorem A [1, *p*.66].** A nontrivial graph is bipartite if and only if all its cycles are even.

## 2. Edge neighborhood graphs

The concept of the neighborhood graph inspires us to introduce the edge neighborhood graph of a graph.

**Definition 1.** Let $G = (V, E)$ be a graph. The edge neighborhood graph $N_e(G)$ of $G$ is the graph with the vertex set $E \cup S$ where $S$ is the set of all open edge neighborhood sets of the edges of $G$ and with two vertices $u$, $v$ in $N_e(G)$ adjacent if $u \in E$ and $v$ is an open edge neighborhood set containing $u$.

**Example 2.** In Figure 1, a graph $G$ and its edge neighborhood graph $N_e(G)$ are shown. For the graph $G$ in Figure 1, $N(e_1) = \{e_2, e_3\}$ $N(e_2) = \{e_1, e_3, e_4\}$, $N(e_3) = (e_1, e_2, e_4\}$ $N(e_4) = \{e_2, e_3\}$ are the open edge neighborhood sets of edges of $G$.




$G$ $\qquad$ $N_e(G)$

**Figure 1:**

We note that the edge neighborhood graph is defined only if $G$ has no isolated vertices.

**Theorem 3.** For any graph $G$ without isolated vertices and without isolated edges, $N_e(G)$ is bipartite.

**Proof:** By definition, no two vertices corresponding edges in $N_e(G)$ are adjacent and no two vertices corresponding to open edge neighborhood sets in $N_e(G)$ are adjacent. Thus $N_e(G)$ has no odd cycles. Thus by Theorem A, $N_e(G)$ is bipartite.

**Remark 4.** If $e$ is an isolated edge of a graph, then $N(e)$ is a null set.

**Theorem 5.** If $G$ is a $(p, q)$ graph without isolated vertices and without isolated edges, then the edge neighborhood graph $N_e(G)$ of $G$ has $2q$ vertices and $\frac{1}{2}\left[\sum d(e_i) + \sum d(N(e_i))\right]$ edges.

**Proof:** Let $G$ be a $(p, q)$ graph without isolated vertices and without isolated edges. Then for each edge $e$ of $G$, the open edge neighborhood set $N(e)$ exists. Therefore $G$ has $q$ open edge neighborhoods sets. Since the vertex set of $N_e(G)$ is the union of the set of edges and the set of open edge neighborhood sets of $G$, it implies that $N_e(G)$ has $2q$ vertices.

In $N_e(G)$, the corresponding vertex $v_i$ of the edge $e_i$ of $G$ contributes $\sum d(e_i)$ edges and corresponding vertex $N(v_i)$ of the open edge neighborhood set $N(e_i)$ of $G$ contributes $\sum d(N(e_i))$ edges. Clearly $\sum d(e_i) = \sum d(N(e_i))$. Thus the number of edges in $N_e(G)$

$$=\frac{1}{2}\left[\sum d\left(e_i\right)+\sum d\left(N\left(e_i\right)\right)\right].$$

**Theorem 6.** If $C_p$ is a cycle with $p \geq 3$ vertices, then

$$\begin{aligned} N_e(C_p) &= 2C_p, \quad \text{if } p \text{ is even} \\ &= C_{2p} \quad \text{if } p \text{ is odd.} \end{aligned}$$

**Proof:** Let $E(C_p) = \{e_1, e_2, \ldots, e_p\}$, $p \geq 3$. Let $N_e(e_i) = Y_i$, $1 \leq i \geq p$.
Then $V(N_e(C_p)) = \{e_1, e_2, \ldots, e_p, Y_1, Y_2, \ldots, Y_p\}$. Consider

$Y_1 = N_e(e_1) = \{e_2, e_p\}$
$Y_2 = N_e(e_2) = \{e_1, e_3\}$
$Y_3 = N_e(e_3) = \{e_2, e_4\}$
. . .
. . .
. . .
$Y_{p-1} = N_e(e_{p-1}) = \{e_{p-2}, e_p\}$
$Y_p = N_e(e_p) = \{e_{p-1}, e_1\}$.

In $N_e(C_p)$, no two corresponding vertices of $e_1, e_2, \ldots, e_p$ are adjacent and no two corresponding vertices of $Y_1, Y_2, \ldots, Y_p$ are adjacent. The adjacencies of the vertices in $N_e(C_p)$ are

$Y_1$ is adjacent with $e_2$ and $e_p$
$Y_2$ is adjacent with $e_3$ and $e_1$
$Y_3$ is adjacent with $e_4$ and $e_2$
: : : :
. . . .
$Y_{p-1}$ is adjacent with $e_{p-2}$ and $e_p$
$Y_p$ is adjacent with $e_{p-1}$ and $e_1$.

**Case 1.** Suppose $p$ is even. Then the adjaceny of vertices of $N_e(C_p)$ is given below.

$Y_1\ e_2\ Y_3\ e_4 \ldots Y_{p-1}\ e_p\ Y_1$
and $Y_2\ e_3\ Y_4\ e_5 \ldots Y_p\ e_{p-1}\ Y_2$.

Since $p$ is even, $Y_1\ e_2\ Y_3\ e_4 \ldots Y_{p-1}\ e_p\ Y_1$ is a cycle with $p$ vertices and $Y_2\ e_3\ Y_4\ e_5 \ldots Y_p\ e_{p-1}\ Y_2$ is also a cycle with $p$ vertices and they are disjoint. Thus $N_e(C_p) = 2C_p$.

**Case 2.** Suppose $p$ is odd. Then the adjacency of the vertices of $N_e(C_p)$ is $e_1\ Y_2\ e_3\ Y_4\ e_5 \ldots e_p\ Y_1\ e_2\ Y_3\ e_4 \ldots Y_p\ e_1$, which is a cycle with $2p$ vertices. Thus $N_e(C_p) = C_{2p}$.

Therefore

$$\begin{aligned} N_e(C_p) &= 2C_p, \text{ if } p \text{ is even} \\ &= C_{2p}, \text{ if } p \text{ is odd.} \end{aligned}$$

**Theorem 7.** If $P_p$ is a path with $p \geq 3$ vertices, then

$$N_e(P_p) = 2\, P_{p-1}.$$

**Proof:** We prove this result by induction on $p$. The result is true for $p$=3 or 4. Assume the result is true for $p = r$. We now prove the result is true for $p = r+1$. Let $G$ be a path with $r$+1 vertices and $v_{r+1}$ be an end vertex of $G$. Then $G_1 = G - v_{r+1}$ is a path with $r$ vertices and $v_r$ is an endvertex of $G_1$. By hypothesis, $Ne(G_1) = 2P_{r-1}$. In $G$, $e_r$ is an end edge. Then

$N(e_r) = e_{r-1}$. Clearly, in $N_e(G)$, the corresponding vertex of $N(e_r)$ is adjacent with the corresponding vertex of $e_{r-1}$ in one component and the corresponding vertex of $e_r$ is adjacent with the corresponding vertex of $N(e_{r-1})$ in another component. Hence $N_e(G) = 2P_r$. Thus the result is true for $p = r+1$. This completes the proof.

**Theorem 8.** $N_e(G) = \overline{K}_p$ if and only if $G = pK_2$, $p \geq 1$.

**Proof:** Suppose $G = pK_2$. Then each component of $G$ is an edge $e$. By Remark 4, $N(e)$ is a null set. Clearly $N_e(G) = \overline{K}_p$.

Conversely suppose $N_e(G) = \overline{K}_p$. We now prove that $G = pK_2$. Assume there exists a component of $G$ which has at least 2 edges, say $uv = e_1$, and $vw=e_2$. Then $N(e_1)$ and $N(e_2)$ are nonempty open edge neighborhood sets of edges $e_1$ and $e_2$ respectively. Thus $N_e(G)$ contains an edge. Thus $N_e(G) \neq \overline{K}_p$, which is a contradiction. Thus each component of $G$ is an edge. Hence $G = pK_2$.

**Theorem 9.** $N_e(G) = 2mP_2$ if and only if $G = mP_3$, $m \geq 1$.

**Proof:** Suppose $N_e(G) = 2mP_2$, $m \geq 1$. Now we prove that $G = mP_3$. Assume $G \neq mP_3$. We consider the following two cases.

**Case 1.** Suppose $G = mP_2$. Then $N_e(G) = mP_1$, which is a contradiction.

**Case 2.** Suppose $G = mG_1$ where $G_1$ is a component of $G$ with at least 4 vertices. Then there exists at least one open edge neighborhood set N containing two or more edges of $G$. By definition, $N$ will form a subgraph $P_3$ in $N_e(G)$, which is a contradiction.

From the above two cases, we conclude that $G = mP_3$.

Conversely suppose $G = mP_3$. If $m = 1$, then $G = P_3$. Clearly each edge $e_i$ forms an open edge neighborhood set $\{e_i\}$. Thus $e_i$ and $\{e_i\}$ are adjacent vertices in $N_e(G_1)$. Since $G_1$ has 2 edges, it implies that $N_e(P_3)= 2P_2$. Hence if $G$ has $m$ components, each of which is $P_3$, then $N_e(mP_3) = 2mP_2$.

**Theorem 10.** If a graph $G$ is $r$-regular, then $N_e(G)$ is $2(r-1)$ -regular.

**Proof:** Suppose $G$ is $r$-regular, $r \geq 1$. Let $V(G)$ be the vertex set of $G$. Then the degree of each vertex is $r$ and $\deg e = \deg u + \deg v - 2 = 2r - 2$, where $u,v \in V(G)$. Thus in $N_e(G)$, $\deg e = \deg N(e) = 2r - 2$. Hence $N_e(G)$ is $2(r-1)$-regular.

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