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On *k*-Minimally Nonouterplanarity of Line Graphs

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Abstract. A graph is said to be embedded in a surface *S* when it is drawn on *S* so that no two lines intersect. A graph is planar if it can be embedded in the plane. In this paper, we obtain results on plane embedding of a graph *G*. Also we present characterization of *k*-minimally nonouterplanar line graphs of certain class of graphs.

***Keywords*:** line graph, embedding, inner point number, planar, *k*-minimally nonouterplanar

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1. Introduction

All graphs considered here are finite, undirected and without loops or multiple lines. We use the terminology of [2].

The line graph of *G*, denoted *L(G)*, is the intersection graph $\Omega(X)$. Thus the points of *L(G)* are the lines of *G*, with two points of *L(G)* are adjacent whenever the corresponding lines of *G* are adjacent.

In [3], the idea of a minimally nonouterplanar graph is introduced. The inner point number *i(G)* of a graph is introduced. The inner point number *i(G)* of a planar graph *G* is the minimum possible number of points not belonging to the boundary of the exterior region in any embedding of *G* in the plane. Obviously *G* is planar if and only if *i(G)*=0. A graph *G* is minimally nonouterplanar if *i(G)*=1, and is *k*-minimally nonouterplanar ($k\geq 2$) if $i(G)=k$. Many other graph valued functions in graph theory were studied, for example, in [4-24].

The following will be useful in the proof of our results.

Theorem A. [1, p.273] *Let G be a graph with p points and q lines. Then*

(i) *The degree in L(G) of a line vw of G is degv+degw–2 ;*

(ii) $L(P_p)\cong P_{p-1}$, *for* $p\geq 1$.

Theorem B. [2, p.72] *A connected graph G is isomorphic to its line graph if and only if it is a cycle.*

Theorem C. [2, p.124] *A graph G has a planar line graph L(G) if and only if G is planar,* $\Delta(G)\leq 4$, *and every point of degree 4 is a cutpoint.*

Theorem D. [2, p.124] *The line graph $L(G)$ of a graph G is outerplanar if and only if $\Delta(G) \leq 3$ and every point of degree 3 is a cutpoint.*

Theorem E. [3] *The line graph of a finite connected graph G is minimally nonouterplanar if and only if G satisfies the following conditions:*

(i) deg $v \leq 4$ for every point v of G

and (ii) G has exactly one point v of degree 4, v lies on at least three blocks of G in which one block has an end point of G and if deg $v_1=3$ for any other point v_1 of G, then v_1 is a cutpoint

or (iii) deg $v \leq 3$ for every point v of G, G has exactly two non cutpoints of degree 3 and these are adjacent.

2. Some important results

We first prove two lemmas, which are useful to prove our main theorem.

A line of a plane graph G is called a boundary line if it is on the boundary of the exterior region, otherwise it is called a nonboundary line.

Lemma 1. *If G is a m-minimally nonouterplanar (p, q) block other than K_2, then it has $(q-p+m)$ nonboundary lines in the plane embedding of G.*

Proof. Suppose G is a m-minimally nonouterplanar (p, q) block other than K_2 and is embedded in the plane. Then partition the points and lines of G as $p=p_1+p_2$ and $q=q_1+q_2$, where p_1 (q_1) be the number of points (lines) lying on the boundary of the exterior region, p_2 be the number of inner points and q_2 be the number of nonboundary lines. Since G is a block, the number of points and lines on the boundary of the exterior region will be equal. That is $p_1=q_1$ and $p_2=m$, since G is m-minimally nonouterplanar. Substitute the value of p_1 and p_2 in p we get, $p=q_1+m$. That is, $q_1=p-m$.

Consider, $q=q_1+q_2$. Then $q = p-m+q_2$ or $q_2=q-p+m$.

Thus G has $(q-p+m)$ nonboundary lines in the plane embedding of G.

Lemma 2. *If G is a m-minimally nonouterplanar (p, q) block with $\Delta(G) \leq 3$ then every nonboundary line in the plane embedding of G corresponds to the inner point of $L(G)$.*

Proof. Suppose G is a m-minimally nonouterplanar (p, q) block with $\Delta(G) \leq 3$ and is embedded in the plane. Let x be any nonboundary line of G. Suppose G has a unique nonboundary line. Then G has exactly two points of degree 3 and these are adjacent. Let v and w be the points of degree 3. Then $x=vw$ is a unique nonboundary line and which is also a chord of a cycle of G. If H is the graph obtained from G by deleting the line x then by Theorem D, $L(H)$ is outerplanar. For the graph G, $L(G)$ is minimally nonouterplanar by Theorem E. It implies that the nonboundary line $x=vw$ of G corresponds to the inner point of $L(G)$.

Suppose G has two nonboundary lines. Then we consider the following cases;

Case 1. Suppose G has exactly two points of degree 3 and are connected by three point-disjoint paths one of which is a path of length 2. Let v and w be the points of degree 3. Then there exists a point u in G such that vu and uw are the two nonboundary lines on the path joining the points v and w. If H is the graph obtained from G by deleting the lines vu and uw then again by Theorem D, $L(H)$ is outerplanar. Also we observe that $L(G)$ is 2-

minimally nonouterplanar with 2 inner points x_1 and x_2, where $x_1=vu$ and $x_2= uw$ are the nonboundary lines of G as shown in Figure 1.

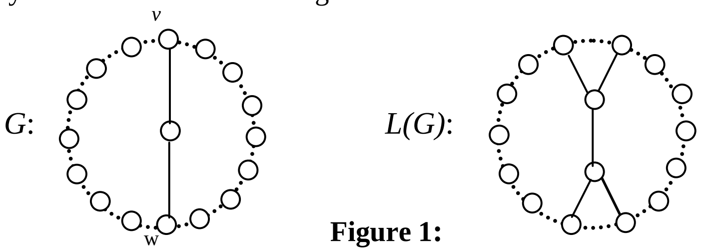

Figure 1:

Case 2. Suppose G has exactly two different pairs of points of degree 3 and the points of each pair are adjacent. Let u, v, u' and v' be the points of degree 3 and let $x_1=uv$ and $x_2=u'v'$ be the nonboundary lines in G. If H is the graph obtained from G by deleting the lines x_1 and x_2 then again by Theorem D, $L(H)$ is outerplanar. Also we observe that $L(G)$ is 2-minimally nonouterplanar with 2 inner points x_1 and x_2 where $x_1=uv$ and $x_2= u'v'$ are the nonbondary lines of G as shown in Figure 2. Thus we conclude that every nonboundary line of G corresponds to an inner point of $L(G)$.

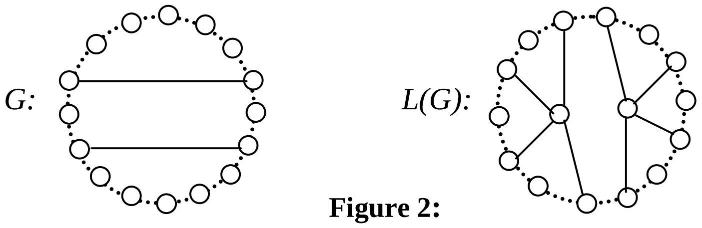

Figure 2:

3. The main results

In the following theorem we characterize those graphs whose line graphs are k-minimally nonouterplanar when the given graph G is a block.

Theorem 3. *Let G be a m-minimally nonouterplanar (p,q) block other than K_2. Then the line graph L(G) is k-minimally nonouterplanar if and only if G satisfies the following conditions;*

(i) $\Delta(G) \leq 3$

and (ii) $q-p+m=k$.

Proof. Suppose G is a m-minimally nonouterplanar (p, q) block other than K_2 and suppose the line graph $L(G)$ is k-minimally nonouterplanar. Then $L(G)$ is planar. By Theorem C, $\Delta(G) \leq 4$ and every point of degree 4 is a cutpoint. Since G is a block, so it has no cutpoint. Thus if $\Delta(G) =4$, then $L(G)$ is nonplanar, a contradiction. Thus $\Delta(G) \leq 3$.

As the graph G is a m-minimally nonouterplanr (p, q) block, by Lemma 1, it has ($q-p+m$) nonboundary lines in the plane embedding of G. Again by Lemma 2, these ($q-p+m$) nonboundary lines of G correspond to the inner points of $L(G)$. It implies that $L(G)$ is ($q-p+m$)-minimally nonouterplanar. Thus the condition $q-p+m=k$ holds.

Conversely, suppose G is a m-minimally nonouterplanar (p, q) block other than K_2 and it satisfies the condition (i) and (ii).

To prove $L(G)$ is k-minimally nonouterplanar, we use induction on k.

Suppose k=0, that is $q–p+m$=0. Then the graph satisfying this condition is a cycle, since G is a block. Then by Theorem B, $L(G)$ is also a cycle which is 0-minimally nonouterplanar. Hence the result is true for k=0.

Suppose k=1, that is $q–p+m$=1. Then G is a block with exactly two points of degree 3 and these are adjacent. Since $\Delta(G)\leq 3$, then by Theorem E, $L(G)$ is 1-minimally nonouterplanar. Hence the result is true for k=1.

Assume the result is true for $k=n$. That is, when $q–p+m=n$, $L(G)$ is n-minimally nonouterplanar.

Now introduce a point on any nonboundary line in the plane embedding of G but not on the boundary line because if we introduce a point on the boundary line then one point and a line is additional and inner points remain same. The additional line formed in G is again a nonboundary line and by Lemma 2, it corresponds to an inner point in $L(G)$. Thus $L(G)$ is (n+1)-minimally nonouterplanar. This completes the proof of the theorem.

The above theorem leads to the following result.

Corollary 4. *Let G be a m-minimally nonouterplanar (p, q) graph such that every component of G is a block other than* K_2*. Then the line graph L(G) is k-minimally nonouterplanar if and only if G satisfies the following conditions;*

(i) $\Delta(G)\leq 3$

and (ii) $q–p+m=k$.

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