Annals of Pure and Applied Mathematics
Vol. 11, No. 2, 2016, 7-10
ISSN: 2279-087X (P), 2279-0888(online)
Published on 10 March 2016
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Average Harmonious Graphs

***N. Adalin Beatress*[1] and *P.B.Sarasija*[2]**

[1]Department of Mathematics, All Saint College of Education, Kaliyakkavilai
Tamil Nadu. E-mail:adalinkamal@yahoo.in

[2]Department of Mathematics, Noorul Islam Centre for Higher Education
Kumaracoil, Tamil Nadu. E-mail: sijavk@gmail.com
[2]Corresponding author

Received 9 January 2016; accepted 4 March 2016

Abstract. Let G(V,E) be a graph with *n* vertices and *m* edges. In this paper, we have introduced and developed the concept of an Average Harmonious Labeling of graphs. A graph *G*(*n*,*m*) is said to be Average harmonious if there exists an injection $f : V \rightarrow \{0, 1, 2, \ldots, m+n\}$ and the induced function $f^* : E \rightarrow \{0, 1, \ldots, (m-1)\}$ defined as

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)+1}{2} \pmod m \ \text{if } f(u)+f(v) \text{ is odd} \\ \dfrac{f(u)+f(v)}{2} \pmod m \ \text{if } f(u)+f(v) \text{ is even} \end{cases}$$

is bijective, the resulting edge labels should be distinct. A graph which admits an Average harmonious labeling is called Average harmonious graph. In this paper we proved that the graph P_n, the graph $C_3 \cup P_n^2$, Y-tree graph,the graph $P_n \odot K_1$ $(n \geq 1)$, the star graph , the graph $C_3 \cup P_n$, the bistar graph $B_{n,n}$, triangular snake graph T_n , and the complete bipartite graph $K_{n,n}$ are Average Harmonious Labeling of graphs.

Keywords: Harmonious labeling, bistar, complete bipartite graph.

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

In this paper, we consider finite, undirected, simple connected graphs. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Harmonious graphs naturally arose in the study by Graham and Sloane [2] of modular versions of additive base problems. Odd and even harmonious graphs were introduced in [3,6]. For a detailed survey on graph labeling we refer to Gallian [1]. We also refer [4, 5, 7,8].

Definition 1.1. A function f is called an Average harmonious labeling of a graph G(V, E) with n vertices and m edges if $f : V \rightarrow \{0, 1, 2, \ldots, m+n\}$ is injective and the induced function $f^*: E \rightarrow \{0, 1, \ldots, (m-1)\}$ is defined as

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)+1}{2} (\bmod m) \text{ if } f(u)+f(v) \text{ is odd} \\ \dfrac{f(u)+f(v)}{2} (\bmod m) \text{ if } f(u)+f(v) \text{ is even} \end{cases}$$

is bijective, the resulting edge labels should be distinct. A graph which admits an average harmonious labeling is called an Average harmonious graph.

In this paper we proved the graph P_n, the graph $C_3 \cup P_n^2$, the graph $P_n \odot K_1$ ($n \geq 1$), Y-tree graph, the star graph, the graph $C_3 \cup P_n$, the bistar graph $B_{n,n}$, triangular snake graph T_n , and complete bipartite graph are Average Harmonious Labeling of graphs.

2. Main results

Theorem 2.1. The path graph P_n ($n \geq 1$) is an Average harmonious graph.

Proof: Let P_n be a path graph with n vertices and m = n-1 edges.

Let $V(P_n) = \{v_i ; 1 \leq i \leq n\}$and $E(P_n) = \{v_i v_{i+1} ; 1 \leq i \leq n-1\}$.

Define an injective function $f : V(P_n) \rightarrow \{0, 1, 2,\dots, m+n\}$ by $f(v_i) = i - 1$,$1 \leq i \leq n$. Then f induces a bijection $f^* : E(P_n) \rightarrow \{0, 1, 2,\dots,m-1\}$. Hence the graph P_n graph is an Average harmonious graph.

Theorem 2.2. The graph $C_3 \cup P_n^2$ ($n \geq 4$) is Average harmonious.

Proof: Let C_3 be a cycle graph with 3 vertices and 3 edges and the graph P_n^2 with n vertices and 2n - 3 edges. Let $V(C_3 \cup P_n^2) = \{v_1, v_2, v_3, u_i, 1 \leq i \leq n\}$ and

$E(C_3 \cup P_n^2) = \{v_1 v_2, v_2 v_3, v_3 v_1, u_i u_{i+1}, 1 \leq i \leq n-1, u_i u_{i+2}, 1 \leq i \leq n-2\}$.

$| V(C_3 \cup P_n^2) | = n+3$ and $| E(C_3 \cup P_n^2) | = m = 2n$

Define an injective function $f : V(C_3 \cup P_n^2) \rightarrow \{0, 1, 2,\dots, 3n +3)\}$ by

$f(v_1) = 0$, $f(v_2) = 2$, $f(v_3) = 4$, $f(u_i) = 2i+1$, $1 \leq i \leq n$.

Then f induces a bijection $f^* : E(C_3 \cup P_n^2) \rightarrow \{0, 1, 2,\dots,2n-1\}$.

Hence the graph $C_3 \cup P_n^2$ graph is an Average harmonious graph.

Theorem 2.3. The graph Y-tree is Average harmonious.

Proof: Consider the graph Y-tree with n vertices and m = n-1 edges.

Let $V(Y) = \{v_1, v_2, \dots,v_n \}$and $E(Y) = \{v_{n-1}v_n, v_{n-2}v_n, v_nv_{n-3}, v_{n-3}v_{n-4},\dots,v_2v_1\}$.

Define an injective function $f : V(Y) \rightarrow \{0, 1, 2,\dots, m+n\}$ by

$f(v_n) = 0$, $f(v_i) = (n-i)$, $1 \leq i \leq n-1$.

Then f induces a bijection $f^*: E(Y) \rightarrow \{0, 1, 2, \dots,n-2\}$. Therefore both edge and vertex labels are distinct. Hence the graph Y-tree is an Average harmonious graph.

Theorem 2.4. The comb graph $P_n \odot K_1$ ($n \geq 1$) is Average harmonious.

Proof: Consider the comb graph with 2n vertices and m = 2 n-1 edges.

Let $V (P_n \odot K_1) = \{ v_i, u_i; 1 \leq i \leq n\}$ and $E (P_n \odot K_1) =\{ v_iv_{i+1} , u_iv_i, 1\leq i \leq n\}$

Define an injective function $f:V (P_n \odot K_1) \rightarrow \{ 0,1,2,\dots\dots m+n \}$ by $f (v_{2i-1}) = 4i-3$, $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$,

$f (v_{2i}) = 4i-2$, $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $f (u_{2i-1}) = 4i - 4$, $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, $f (u_{2i}) = 4i-1$, $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

Then f induces a bijection $f^*: E(P_n \odot K_1) \rightarrow \{0, 1, 2,\ldots,m-1\}$.
Hence we obtain the graph $P_n \odot K_1$ is an Average harmonious graph.

Theorem 2.5. The star graph S_n is an Average harmonious graph for all n.
Proof : Let S_n be a star graph with n vertices and m = n-1 edges.
Consider v as the center vertex. Let $V(S_n) = \{v, v_i\ ; 1 \le i \le n - 1\}$ and $E(S_n) = \{vv_i\ ; 1 \le i \le n-1\}$.
Define $f : V(S_n) \rightarrow \{0, 1, 2,\ldots, m+n\}$ by $f(v) = 1, f(v_1) = 0, f(v_i) = 2i - 1,\ 2 \le i \le n-1$.
Then f induces a bijection $f^* : E(S_n) \rightarrow \{0, 1, 2,\ldots,m-1\}$. Hence, the S_n graph is an Average harmonious graph.

Theorem 2.6. The graph $C_3 \cup P_n$ ($n \ge 2$) is an Average harmonious graph.
Proof: Let P_n be a path with n vertices and (n – 1) edges and C_3 be a cycle with 3 vertices and 3 edges. Let $V(C_3 \cup P_n) = \{v_1, v_2, v_3, u_i, 1 \le i \le n\}$ and
$E(C_3 \cup P_n) = \{v_1 v_2, v_2 v_3, v_3 v_1, u_i u_{i+1}, 1 \le i \le n-1\}$,
$|V(C_3 \cup P_n)| = n+3$ and $|E(C_3 \cup P_n)| = m = n+2$
Define an injective function $f : V(C_3 \cup P_n) \rightarrow \{0, 1, 2,\ldots, 2n + 3\}$ by $f(v_1) = 1$, $f(v_2) = 3$, $f(v_3) = 5$, $f(u_1) = 0$ $f(u_{2i}) = 2i$, $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, $f(u_{2i+1}) = 2i+5$, $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$
Then f induces a bijection $f^*: E(C_3 \cup P_n) \rightarrow \{0, 1, 2,\ldots,m-1\}$. Hence the $C_3 \cup P_n$ graph is Average harmonious.

Theorem 2.7. The bistar graph $B_{n,n}$ is Average harmonious.

Proof: Let the bistar graph $B_{n,n}$ with 2 n + 2 vertices and 2 n +1edges.
Let $V(B_{n,n}) = \{u, v, u_i, v_i, 1 \le i \le n\}$ and $E(B_{n,n}) = \{u_i u, v_i v, 1 \le i \le n, uv\}$
Define an injective function $f : V(B_{n,n}) \rightarrow \{0, 1, 2,\ldots, 4n+3)\}$ by $f(u_1) = 1$, $f(u) = 0$, $f(u_i) = 4i - 1,\ 2 \le i \le n$. $f(v) = 3$, $f(v_i) = 4i – 2,\ 1 \le i \le n$
Then f induces a bijection $f^* : E(B_{n,n}) \rightarrow \{0, 1, 2,\ldots,m-1\}$.
The edge labels are given below: $f^*(uu_1) = 1, f^*(u_i u) = 2i \pmod{2n+1}$, $2 \le i \le n$. $f^*(uv) = 2$, $f^*(v_i v) = (2i + 1) \pmod{2n+1}$ $1 \le i \le n$. Hence the bistar graph is an Average harmonious graph.

Theorem 2.8. A triangular snake T_n admits Average harmonious labeling.
Proof: Let $V(T_n) = \{u_i, 1 \le i \le n; v_i, 1 \le i \le n-1\}$

$$E(T_n) = \begin{cases} u_i u_{i+1} \ \text{if } 1 \le i \le n-1 \\ u_i v_i \ \text{if } 1 \le i \le n-1 \\ u_{i+1} \ v_i \ \text{if } 1 \le i \le n-1 \end{cases}$$

Define an injective function $f : V(T_n) \rightarrow \{0\ 1,\ldots,m+n\}$ by $f(u_i) = (i – 1),\ 1 \le i \le n$, $f(v_i) = 2n + 3(i-1),\ 1 \le i \le n-1$.

The induced function $f^* : E(T_n) \rightarrow \{0,1,\ldots,m-1\}$ is a bijection. It can be observed that the triangle snake graph is Average harmonious graph.

Example 2.9. The following figure gives an Average harmonious labeling for the triangle snake T_6.

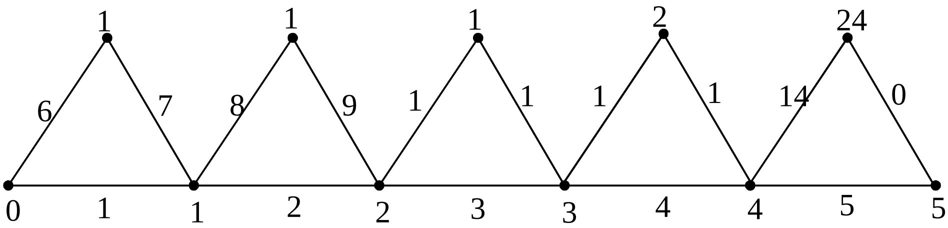

Figure 1: Triangle snake graph

Theorem 2.10. The complete bipartite graph $K_{n,n}$ is an Average harmonious graph.
Proof: A complete bipartite graph is a simple bipartite graph with bipartition of the vertex set V into X and Y in which each vertex of X is joined to each vertex of Y. Let the vertices of the set X be ui ,$1 \leq i \leq n$ and the vertices of the set Y be v_i $1 \leq i \leq n$
The edges are u_iv_i, $1 \leq i \leq n$. Hence there are 2n vertices and n^2 edges.
Define an injective function $f : V \rightarrow \{ 0,1,\ldots,4n)\}$ by $f(u_i) = 2i-1$, $1 \leq i \leq n$,$f(v_1) = 0$ and $f(v_i) = 2(i-1)\, n$, $2 \leq i \leq n$.

The induced function $f^* : E(K_{n,n}) \rightarrow \{ 0,1,\ldots.,n^2-1\}$ is a bijection. It can be observed that the complete bipartite graph $K_{n,n}$ is an Average harmonious graph.

REFERENCES

1. J.A.Gallian, A dynamic survey of graph labeling, *The Electronics J. Combinatorics*, 16 (2014) .
2. R.L.Graham and N.J.A.Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Meth.*, 1 (1980) 382-404.
3. Z-H Liang and Z-L Bai, On the odd harmonious graphs with applications, *J. Appl. Math. Comput.*, 29 (2009) 105-116.
4. B.Liu and X.Zhang, On harmonious labeling of graphs, *Ars. Combin.*, 36 (1993) 315 – 326.
5. N.Adalin Beatress and P.B.Sarasija, Even-odd harmonious graphs, *International Journal of Mathematics and Soft Computing*, 5(1) (2015) 23-29.
6. P.B.Sarasija and R.Binthiya, Even harmonious graphs with applications, *Intern. Journal of Computer Science and Information Security*, 9(7) (2011)161-163.
7. T.Tharmaraj and P.B.Sarasija, Adjacent edge graceful graphs, *Annals of Pure and Applied Mathematics*, 6(1) (2014) 25-35.
8. T.Tharmaraj and P.B.Sarasija, Some adjacent edge graceful graphs, *Annals of Pure and Applied Mathematics*, 6(2) (2014) 160-169.