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The Disjoint Total Domination Number of a Graph

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Abstract: The disjoint total domination number of a graph G is the minimum cardinality of the union of two disjoint total dominating sets in G. We also consider an invariant the minimum cardinality of the disjoint union of a dominating set and a total dominating set. In this paper, we initiate a study of these parameters.

Keywords: inverse total dominating set, disjoint total dominating sets, inverse total domination number, disjoint total domination number.

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

We consider graphs $G = (V, E)$ with vertex set V and edge set E which are finite, undirected without loops and multiple edges. Any undefined term here may be found in Kulli [1, 2].

For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ of S is defined by $N(S) = \bigcup_{v\in S} N(v)$, for all $v \in S$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ adjacent to a vertex in D, that is $N[D] = V$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set D of G. A γ-set is a minimum dominating set.

Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D' of G, then D' is called an inverse dominating set of G with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cardinality of an inverse dominating set of G. This concept was introduced by Kulli and Sigarkanti in [3]. Many other inverse domination parameters were studied, for example, in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

The disjoint domination number $\gamma\gamma(G)$ is defined as follows: $\gamma\gamma(G) = \min \{|D_1|+ |D_2| : D_1$ and D_2 are disjoint dominating sets of $G\}$. This concept was introduced by Hedetniemi *et al.* in [14]. Many other disjoint domination parameters were studied, for example, in [5, 6, 11, 15].

A set $D \subseteq V$ is a total dominating set of G if every vertex in V is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G.

Let $D \subseteq V$ be a minimum total dominating set of G. If $V - D$ contains a total dominating set D' of G, then D' is called an inverse total dominating set with respect to D. The inverse total domination number $\gamma_t^{-1}(G)$ of G is the minimum cardinality of an inverse total dominating set of G. This concept was introduced by Kulli and Iyer in [16] and was studied, for example, in [17, 18].

Two graphs G_1 and G_2 have disjoint vertex sets V_1, V_2 and edge sets E_1, E_2 respectively. Their union is denoted by $G_1 \cup G_2$ and it has $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Their join is denoted by $G_1 + G_2$ and it consists of $G_1 \cup G_2$ and all edges joining every vertex of V_1 with every vertex of V_2. The corona of two graphs G_1 and G_2 is the graph $G = G_1 \text{ o } G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

In this paper, we initiate a study of the disjoint total domination number and establish some results of this parameter.

1. Disjoint total domination number

Definition 1. The disjoint total domination number $\gamma_t\,\gamma_t(G)$ of a graph G is defined as follows: $\gamma_t\gamma_t(G) = \min\{|D_1|+|D_2|: D_1, D_2$ are disjoint total dominating sets of $G\}$, (see [2]).

We say that two disjoint total dominating sets, whose union has cardinality $\gamma_t\gamma_t(G)$, is a $\gamma_t\gamma_t$-pair of G.

Note that not all graphs have disjoint total domination number. For example, each cycle C_{2n+1}, $n \geq 1$ does not have two disjoint total dominating sets.

Theorem 2. If a graph G has a γ_t^{-1}-set, then

$$2\,\gamma_t(G) \leq \gamma_t\gamma_t(G) \leq \gamma_t(G) + \gamma_t^{-1}(G) \leq p.$$

We also consider an invariant the minimum cardinality of a disjoint union of a dominating set D and a total dominating set D' and it is denoted by $\gamma\gamma_t(G)$. We call such a pair of dominating sets (D, D'), a $\gamma_t\gamma_t$-pair. A $\gamma\gamma_t$-pair can be found by letting D' be any total dominating set, and then noting that the complement $V - D'$ is a dominating set. Thus $V - D'$ contains a minimal dominating set D.

Note that not all graphs have a $\gamma\gamma_t$-pair. For example, the path P_3 does not have a $\gamma\gamma_t$-pair.

Proposition 3. If both $\gamma\gamma_t$-pair and $\gamma_t\gamma_t$-pair exist, then

$$\gamma\gamma(G) \leq \gamma\gamma_t(G) \leq \gamma_t\gamma_t(G).$$

Proposition 4. If K_p is a complete graph with $p \geq 4$ vertices, then

$$2\gamma_t(K_p) = \gamma_t\gamma_t(K_p) = 4.$$

Proposition 5. If $K_{m,n}$ is a complete bipartite graph with $2 \leq m \leq n$, then

$$2\,\gamma_t(K_{m,n}) = \gamma_t\gamma_t(K_{m,n}) = 4.$$

Proposition 6. If C_{4n} is a cycle with $4n$ vertices, $n \geq 1$, then

$$2\,\gamma_t(C_{4n}) = \gamma_t\gamma_t(C_{4n}) = 4n.$$

Proposition 7. For the cycle C_4,

$$\gamma\gamma(C_4) = \gamma_t\gamma_t(C_4).$$

Proposition 8. If $K_{m,n}$, is a complete bipartite graph with $2 \leq m \leq n$,

$$\gamma\gamma(K_{m,n}) = \gamma_t\gamma_t(K_{m,n}) = 4.$$

A graph G is called $\gamma_t\gamma_t$ -minimum if $\gamma_t\gamma_t(G) = 2\gamma_t(G)$. Similarly, a graph G is called $\gamma_t\gamma_t$-maximum if $\gamma_t\gamma_t(G) = p$.

One can see that the complete graph K_4 and the cycle C_4 are $\gamma_t\gamma_t$-maximum.

The following classes of graphs are $\gamma_t\gamma_t$-minimum.

(i) The complete graphs K_p, $p\geq 4$, are $\gamma_t\gamma_t$-minimum.
(ii) The complete bipartite graphs $K_{m,n}$, $2 \leq m \leq n$ are $\gamma_t\gamma_t$-minimum.
(iii) All cycles C_{4n}, $n\geq 1$, are $\gamma_t\gamma_t$ -minimum.

Theorem 9. A nontrivial tree does not contain two disjoint total dominating sets.

Proof: Suppose $T = P_2$. Clearly it does not contain two disjoint total dominating sets.

Suppose T is a tree with $p \geq 3$ vertices. Let u be an end vertex and v be the support of u. Then there exists a vertex w such that w is adjacent to v. Let D be a γ_t-set of T. We consider the following two cases.

Case 1. Suppose $u, v \in D$. Since w is not adjacent to u, it implies that $V - D$ does not contain another γ_t-set.

Case 2. Suppose $v, w \in D$. The vertex u is not adjacent to any vertex of $V - D$. Thus $V - D$ does not contain another γ_t-set.

From the above two cases, we conclude that T does not contain two disjoint γ_t-sets.

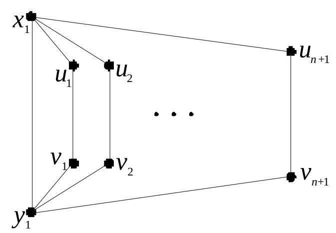

Figure 1:

Theorem 10. For each integer $n \geq 1$, there exists a connected graph G such that $\gamma_t^{-1}(G) - \gamma_t(G) = 2n$ and $|V(G)| = \gamma_t(G) + \gamma_t^{-1}(G)$.

Proof: Let $n \geq 1$. Consider the graph G with $2n+4$ vertices as in Figure 1. Then $D = \{x_1, y_1\}$ is a total dominating set in G, which is minimum. Therefore $\gamma_t(G) = 2$. Since u_i, v_i are adjacent in G for $i = 1, 2, \ldots, n+1$, it implies that $D = V(G) - \{x_1, y_1\}$ is the unique minimum total dominating set in G. Therefore $\gamma_t^{-1}(G) \leq |D| = 2n+2$. Since $N_G[S] \neq V(G)$ for all proper subsets of S of D, it implies that $\gamma_t^{-1}(G) = |D| = 2n+2$. Hence $\gamma_t^{-1}(G) - \gamma_t(G) = 2n$ and also $|V(G)| = \gamma_t(G) + \gamma_t^{-1}(G)$.

Theorem 11. For each integer $n \geq 1$, there exists a connected graph G such that $\gamma_t(G) + \gamma_t^{-1}(G) - \gamma_t\gamma_t(G) = 2n$.

Proof: Consider the graph G as in Figure 2 obtained by adding to the corona $C_4 \circ C_4$ $2n$ vertices $x_1, y_1, x_2, y_2, ..., x_n, y_n$ and the edges x_iu_j, y_iu_j, x_iy_i, $i = 1,2, \ldots, n$, $j = 1, 2, 3, 4$. Then $\{u_1, u_2, u_3, u_4\}$ is the unique minimum total dominating set in G and

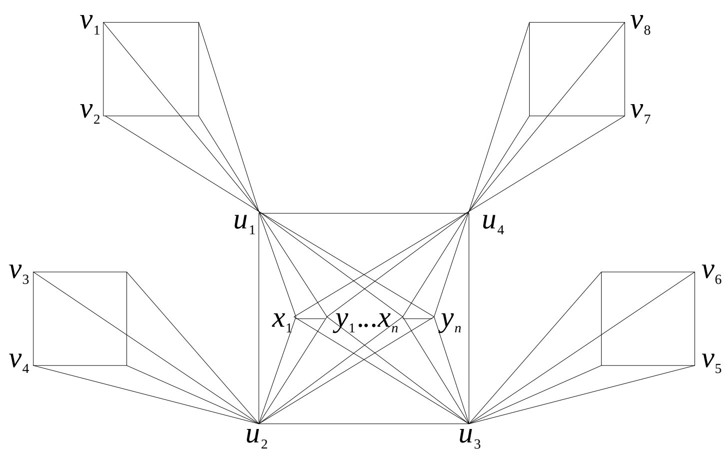

Figure 2: Graph G

$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \cup \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ is a γ_t^{-1}-set in G. Thus $\gamma_t(G) = 4$ and $\gamma_t^{-1}(G) = 8 + 2n$. Also the sets $D_1 = \{u_1, u_2, v_5, v_6, v_7, v_8\}$ and $D_2 = \{u_3, u_4, v_1, v_2, v_3, v_4\}$ constitute a $\gamma_t\gamma_t$-pair in G. Hence $\gamma_t\gamma_t(G) = |D_1|+|D_2| = 12$. Therefore $\gamma_t(G) + \gamma_t^{-1}(G) - \gamma_t\gamma_t(G) = 2n$.

Corollary 12. The difference $\gamma_t(G) + \gamma_t^{-1}(G) - \gamma_t\gamma_t(G)$ can be made arbitrarily large.

We consider pair of disjoint total dominating sets in the join graphs.

Proposition 13. If a γ_t^{-1}-set exists in a graph G, then

$$\gamma_t\gamma_t(G+K_2) = 2 + \gamma_t(G) = 2 + \gamma_t^{-1}(G + K_2).$$

Proposition 14. Let G and H be nontrivial graphs. If a γ_t^{-1}(G+H) exists, then $\gamma_t\gamma_t(G+H) = 4$.

Proof: In $G + H$, each vertex of G is adjacent to every vertex of H and vice versa. Thus pick $u \in G$, $v \in H$ and choose $x \in V(G) - \{u\}$ and $y \in V(H) - \{v\}$. Then $D = \{u, v\}$ and $S = \{x, y\}$ are disjoint γ_t-sets in $G + H$. Thus $\gamma_t^{-1}(G+H)=2$ and $\gamma_t(G+H)=2$. Thus

$$2\gamma_t(G+H) = \gamma_t\gamma_t(G+H) = \gamma_t(G + H) + \gamma_t^{-1}(G+H) = 4.$$

Corollary 15. Let G and H be nontrivial graphs and p be the number of vertices of $G + H$. Then $\gamma_t(G+H) + \gamma_t^{-1}(G+H) = p$ if and only if $G = K_2$ and $H = K_2$.

Some open problems

Problem 1. Characterize the class of $\gamma_t\gamma_t$- minimum graphs.
Problem 2. Characterize the class of $\gamma_t\gamma_t$-maximum graphs.
Problem 3. Under what conditions does $\gamma_t\gamma_t(G)$ exist?
Problem 4. When is $\gamma\gamma(G) = \gamma_t\gamma_t(G)$?
Problem 5. When is $\gamma\gamma(G) = \gamma\gamma_t(G)$?
Problem 6. When is $\gamma\gamma_t(G) = \gamma_t\gamma_t(G)$?
Problem 7. Obtain an upper bound for $\gamma_t\gamma_t(G) + \gamma_t\gamma_t(\overline{G})$.

REFERENCES

1. V.R.Kulli, *College Graph Theory,* Vishwa International Publications, Gulbarga, India (2012).
2. V.R.Kulli, *Theory of Domination in Graphs*, Vishwa International Publications, Gulbarga, India (2010).
3. V.R.Kulli and S.C. Sigarkanti, Inverse domination in graphs, *Nat. Acad. Sci. Lett.*, 14 (1991) 473-475.
4. V.R.Kulli, *Inverse total edge domination in graphs*. In Advances in Domination Theory I, V.R.Kulli ed., Vishwa International Publications, Gulbarga, India (2012) 35-44.
5. V.R.Kulli, Inverse and disjoint neighborhood total dominating sets in graphs, *Far East J. of Applied Mathematics*, 83(1) (2013) 55-65.
6. V.R.Kulli, Inverse and disjoint neighborhood connected dominating sets in graphs, *Acta Ciencia Indica*, Vol.XL M, (1) (2014) 65-70.
7. V.R.Kulli and R.R.Iyer, Inverse vertex covering number of a graph, *Journal of Discrete Mathematical Sciences and Cryptography,* 15(6) (2012) 389-393.
8. V.R.Kulli and B.Janakiram, On n-inverse domination number in graphs, *A.N. International Journal of Mathematics and Information Technology,* 4 (2007) 33-42.
9. V.R.Kulli and M.B. Kattimani, The inverse neighborhood number of a graph, *South East Asian J. Math. and Math. Sci.* 6(3) (2008) 23-28.
10. V.R.Kulli and M.B. Kattimani, *Inverse efficient domination in graphs*. In Advances in Domination Theory I, V.R.Kulli, ed., Vishwa International Publications, Gulbarga, India, (2012) 45-52.
11. V.R.Kulli and Nirmala R.Nandargi, *Inverse domination and some new parameters*. In Advances in Domination Theory I, V.R.Kulli, ed., Vishwa International Publications, Gulbarga, India, (2012) 15-24.
12. V.R.Kulli, Inverse domination and inverse total domination in digraphs, *Inter. J. of Advanced Research in Computer Science and Technology* 2(1) (2014).
13. V.R.Kulli and N.D.Soner, Complementary edge domination in graphs, *Indian J. Pure Appl. Math.*, 28 (1997) 917-920.

14. S.M.Hedetnimi, S.T.Hedetniemi, R.C.Laskar, L.Markus and P.J.Slater, Disjoint dominating sets in graphs, Discrete Mathematics, *Ramanujan Math. Soc. Lect. Notes Series*, 7 (2008) 87-100.
15. V.R.Kulli, The disjoint vertex covering number of a graph, *International J. of Math. Sci. and Engg. Appls.* 7(5) (2013) 135-141.
16. V.R.Kulli and R.R.Iyer, Inverse total domination in graphs, *Journal of Discrete Mathematical Sciences and Cryptography*, 10(5) (2007) 613-620.
17. V.R.Kulli, Graphs with equal total domination and inverse total domination numbers, *International Journal of Mathematics and its Applications*, 4 (2016).
18. V.R.Kulli, Inverse total domination in the corona and join of graphs, *Journal of Computer and Mathematical Sciences*, 7(2) (2016) 61-64.