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Total Dominating Color Transversal Number of Graphs

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Abstract. In this paper we introduce a new concept called Total Dominating Color Transversal number of a graph. We find this number for path graph and cycle graph. We also find interesting relation between this number and Total Domination number and Chromatic number. Along with it we also find some other results related to this number

Keywords: Total Dominating Color Transversal number, χ – Partition of a graph and Transversal.

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1. Introduction

We begin with simple, finite, connected and undirected graph without isolated vertices. In [1], Manoharan, introduced the concept of dominating color transversal sets of a graph. His work was basically on domination theory. This gave us the idea of working on Total Domination theory. We know that proper coloring of vertices of graph G partitions the vertex set V of G into equivalence classes (also called the color classes of G). Using minimum number of colors to properly color all the vertices of G yields χ equivalence classes. Total dominating color transversal set of graph G is a total dominating set with the extra property that it is also transversal of some such χ - Partition of G.

2. Definitions

Definition 2.1. [2] (Total dominating set) Let G = (V, E) be a graph. Then a subset S of V (G) is said to be a Total Dominating Set of G if for each v ∈ V, v is adjacent to some vertex in S.

Definition 2.2. [2] (Minimum total dominating set) Let G = (V, E) be a graph. Then a total dominating set s is said to be a minimum total dominating set of G if its cardinality is minimum among all total dominating sets of G. Here S is calledγ_t −set and its cardinality, denoted by γ_t(G) or just by γ_t, is called the Total Domination number of G.

Definition 2.3. (χ -partition of a graph) Proper coloring of vertices of a graph G, by using minimum number of colors, yields minimum number of independent subsets of vertex set of G called equivalence classes (also called color classes of G). Such a partition of a vertex set of G is called a χ - partition of the graph G.

Definition 2.4. (Transversal of a χ - partition of a graph) Let G = (V, E) be a graph with χ – Partition $\{V_1, V_2, ..., V_\chi\}$. Then a set $S \subset V$ is called a Transversal of this χ – Partition if $S \cap V_i \neq \emptyset$, $\forall$ i $\in \{1, 2, 3, ..., \chi\}$.

Definition 2.5. (Total dominating color transversal set) Let G = (V, E) be a graph. Then $S \subset V$ is called a total dominating color transversal set of G if it is both total dominating set as well as transversal of at least one $\chi -$ partition of G.

Definition 2.6. (Minimum total dominating color transversal set) Let G = (V, E) be a graph. Then $S \subset V$ is called a minimum total dominating color transversal set of G if its cardinality is minimum among all total dominating color transversal sets of G. Here S is calledγ_{tsdt} $-$Set and its cardinality, denoted by by γ_{tstd}(G) or by just γ_{tstd}, is called the total dominating color transversal number of G.

Definition 2.7. [3] (Total global dominating set) Let G = (V, E) be a graph with δ (G) $\geq$ 1 and δ ($\overline{\mathrm{G}}$) $\geq$ 1. A total dominating set D of a graph G is a total global dominating set if D is also a total dominating set of $\overline{\mathrm{G}}$. (Here $\overline{\mathrm{G}}$ is the complement graph of graph G.)

Definition 2.8. [3] (Minimum total global dominating set) Let G = (V, E) be a graph with δ (G) $\geq$ 1 and δ ($\overline{\mathrm{G}}$) $\geq$ 1. Then $S \subset V$ is called a minimum total global dominating set of G if its cardinality is minimum among all the total global dominating sets of G. Here S is called γ_{tg} – set and its cardinality, denoted by γ_{tg}(G) or just by γ_{tg}, is called the total domination number of G.

3. Main results

Result 3.1. For any graph G, (i) $1 \leq \omega \leq \chi \leq \gamma_{\mathrm{tstd}}$ and (ii) $\gamma_{\mathrm{t}} \leq \gamma_{\mathrm{tstd}}$. ($\omega$ and χ are, respectively, clique number and chromatic number of graph G.)

Theorem 3.2. Let G = (V, E) be a graph with chromatic number χ and minimum total dominating set S. If $< S >$ contains a contains complete sub graph of order χ then γ_{tstd} (G)= γ_{t}(G). (where $< S >$ is the induced sub graph of G formed by the vertices in S).

Proof: Suppose S is a minimum total dominating set of G and $<$ S $>$ contains complete sub graph say H of order χ. Note that all the vertices of H must be assigned distinct χ colors and hence S is a transversal of every $\chi -$partition of G. Hence S is a total dominating color transversal set of G. Therefore $\gamma_{tstd}(G) \leq \gamma_t(G)$. As $\gamma_t(G) \leq \gamma_{tstd}(G)$, γ_{tstd} (G)= γ_t(G).

Theorem 3.3. If χ(G) = 2 then γ_{tstd}(G) = γ_t (G).

Proof: Given that $\boldsymbol{\chi}$(G) = 2. We know that if S is a minimum total dominating set of G then <S> contains complete sub graph of order 2. Hence by Theorem 3.2, γ_{tstd}(G) = γ_{t} (G).

Corollary 3.4. For n $\geq$ 2, $\gamma_{\mathrm{tstd}}(P_n) = \gamma_{\mathrm{t}}(P_n)$ and γ_{tstd}(T) = γ_{t} (T) (where $\mathrm{P_n}$ and T, are respectively, path graph with n vertices and Tree graph)

Proof: $\mathbf{P_n}$ and T are bipartite graph and by Theorem 3.3.

Result 3.5. [2] For n ≥ 2, $\boldsymbol{\gamma_t}$ $(\boldsymbol{P_n})$ $= \frac{\boldsymbol{n}}{\mathbf{2}}$, if n≡0mod(4)

$= \frac{\boldsymbol{n+2}}{\mathbf{2}}$, if n≡2mod(4)

$= \frac{\boldsymbol{n+1}}{\mathbf{2}}$, otherwise

Result 3.6. [2] For n ≥ 3,$\boldsymbol{\gamma_t}$ $(\boldsymbol{C_n})$ = $\boldsymbol{\gamma_t}$ $(\boldsymbol{P_n})$.

Now we will prove that the total dominating color transversal number and total domination number are equal for a cycle graph C_n, n ≥ 4.

Theorem 3.7. For n ≥ 4,γ_{tstd} (C_n) = $\gamma_t(C_n)$ = $\frac{\mathbf{n}}{\mathbf{2}}$**,** if n≡0mod(4)

$= \frac{\mathbf{n+2}}{\mathbf{2}}$, if n≡2mod(4)

$= \frac{\mathbf{n+1}}{\mathbf{2}}$, otherwise

Proof: We first note that cycle with even vertices is bipartite and otherwise it is tripartite. Divide vertices of $\mathbf{C_n}$ into groups of four like $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$, $\{\mathbf{v_5}, \mathbf{v_6}, \mathbf{v_7}, \mathbf{v_8}\}$,….. where the last group may contain one, two, three or four vertices.

Case (I): n≡0 mod (4) or n≡2 mod (4).
In such case Last group has four vertices or two vertices. So cycle $\mathbf{C_n}$ will have even number of vertices and hence $\mathbf{C_n}$ will be bipartite. Hence by Theorem 3.3 and by γ_t (C_n) = γ_t (P_n), We have for n ≥ 4,

$\boldsymbol{\gamma_{tstd}}$ $(\mathbf{C_n})$ = $\boldsymbol{\gamma_t}(\mathbf{C_n})$ = $\frac{n}{2}$, if n≡0mod(4)

$= \frac{\boldsymbol{n+2}}{\mathbf{2}}$, if n≡2mod(4).

Case (II): n≡1mod (4)
Here we first note that $\boldsymbol{\gamma_{tstd}}$ $(\mathbf{C_n}) \geq \boldsymbol{\gamma_t}(\mathbf{C_n}) = \frac{\boldsymbol{n+1}}{\mathbf{2}}$.

In this case last group has one vertex. So cycle $\mathbf{C_n}$ will have odd number of vertices and hence $\mathbf{C_n}$will be tripartite. Select middle two vertices from each group of four vertices except second last group and from second last group select last three vertices. The resultant set, say S, will be a total dominating set with cardinality is $\frac{\boldsymbol{n-5}}{\mathbf{2}} + 3 = \frac{\boldsymbol{n+1}}{\mathbf{2}}$.
Consider the $\chi -$ coloring of vertices of each group by using 1,2 and 3 colors as
{1, 2,1, 3},{2,1,2,3},{2,1,2,3},{2,1,2,3} (2)

Note that the set S will be a transversal of $\chi -$ partition of G formed by such $\chi -$ coloring of G and this set S is a minimum total dominating color transversal set of G with cardinality $\frac{\boldsymbol{n+1}}{\mathbf{2}}$. Hence $\boldsymbol{\gamma_{tstd}}$ $(\mathbf{C_n})$ = $\frac{\mathbf{n+1}}{\mathbf{2}}$, if n≡1mod (4).

Case (III): n≡3 mod (4)

Here we first note that $\boldsymbol{\gamma_{tstd}}$ $(\mathbf{C_n}) \geq \boldsymbol{\gamma_t}(\mathbf{C_n}) = \frac{\mathbf{n+1}}{\mathbf{2}}$.

In this case cycle C_n will be tripartite. Select middle two vertices from each group except from last group and from last group select last two vertices. The resultant set, say S, will be a total dominating set with cardinality $\frac{n-3}{2}$ + 2 = $\frac{n+1}{2}$. Consider the $\boldsymbol{\chi}-$ coloring of vertices of each group by using 1, 2 and 3 colors as {1,2,1,3}, {2,1,2,3}, {2,1,2,3}, ….. {1,2,3}. Note that the S will be a Transversal of $\boldsymbol{\chi}-$ partition of G formed by such $\boldsymbol{\chi}-$ Coloring of G and this set S is a minimum total dominating color transversal set of G with cardinality $\frac{n+1}{2}$.
Hence $\boldsymbol{\gamma_{tstd}}$ $(\mathbf{C_n}) = \frac{\mathbf{n+1}}{\mathbf{2}}$, if n≡3mod(4).

Hence the theorem.

Theorem 3.8. If $\boldsymbol{\gamma_{tstd}}$ (G) = 2 then G is bipartite.
Proof: We have χ (G)≤ $\boldsymbol{\gamma_{tstd}}$ (G) = 2.Henceχ (G) =1 or χ (G)= 2.χ (G) = 1 is not possible as G is a connected graph with δ (G) ≥ 1. Hence χ (G)= 2. i.e. G is bipartite.

Remark 3.9. Converse of Theorem 3.8 is not true in general. For example consider the path graph P_5. χ (P_5) = 2 but $\boldsymbol{\gamma_{tstd}}$ (P_5) = 3.

Theorem 3.10. Let G be a bipartite graph with bipartition X and Y. Then $\boldsymbol{\gamma_{tstd}}$ (G) = 2 iff **there exists** x ∈ X and y ∈ Y such that N (x)= Y and N (y)= X.
Proof: Suppose $\boldsymbol{\gamma_{tstd}}$(G) = 2. So we have $\boldsymbol{\gamma_t}$ (G)= 2. Hence there exists x, y ∈ V such that S = {x, y} totally dominates all the vertices of G. Note that x and y have to be adjacent. So they belong to different color classes. Without loss of generality assume that x ∈ X and y ∈Y. x and y cannot dominate the other vertices of their respective color classes. Now since x and y totally dominate all the vertices of G, x dominates all the vertices of Y and y dominates all the vertices of X. Hence N (x)= Y and N (y)= X.
Conversely, suppose that **there exists** x ∈ X and y∈Y such that N(x)= Y and N (y)= X. Trivially {x, y} totally dominates all the vertices of G as x and y, respectively, dominates all the vertices of Y and X. Hence {x, y} is a minimum totally dominating color transversal set and hence $\boldsymbol{\gamma_{tstd}}$(G) = 2.

Theorem 3.11. If $\boldsymbol{\gamma_t}$(G)= 2 then $\boldsymbol{\gamma_{tstd}}$(G) = χ(G).
Proof: Consider graph G with $\boldsymbol{\gamma_t}$(G)= 2and $\{v_1, v_2\}$ be the minimum total dominating set of G. One must note that this v_1 and v_2 are adjacent and hence they are in different color classes of everyχ- partion of G. Select one vertex from each remainingχ- 2 color classes (the color classes in which v_1 and v_2 are not present). The resultant set will be minimum total dominating color transversal set with cardinality χ. Hence $\boldsymbol{\gamma_{tstd}}$(G)= χ(G).

Corollary 3.12. For Complete k- partite graph G,$\boldsymbol{\gamma_{tstd}}$ (G) = χ (G) (k ≥ 2).

Proof: For complete k- partite graph $\boldsymbol{\gamma_t}$= 2. Hence by Theorem 3.11, $\boldsymbol{\gamma_{tstd}}$ (G)= χ (G).

Corollary 3.13. $\boldsymbol{\gamma_{tstd}}(\mathbf{K_n})$= n.

Theorem 3.14. [3] A total dominating set D of G = (V, E) is a total global dominating set if and only if for each vertex v ∈ V there exists u ∈ D such that v is not adjacent to u.

Theorem 3.15. [3] Let G = (V, E) be a graph with diam(G) ≥ 5.Then D is total dominating Set of G if and only if it is a total global dominating set (diam (G) is the diameter of the graph G).

Remark 3.16. Since the next Theorem 3.17 and 3.18 deals with global total domination we accept that the graph G and its complement$\overline{G}$ have no isolated vertex.

Theorem 3.17. A total dominating color transversal set D of G = (V, E) is a total global dominating set if and only if for each vertex v ∈ D there exists u ∈ D such that v is not adjacent to u.

Proof: Assume that D is a total dominating color transversal set of G and for each vertex v ∈ D there exists u ∈ D such that v is not adjacent to u. So by theorem 3.14, it is enough to prove that for each v ∈ V \D there exists some u ∈ D such that u and v are not adjacent. Suppose there exists some v ∈ V \D such that v is adjacent to every vertex in D. Then as D is a total dominating color transversal set of G, it has at least one vertex from all the colors classes, for some χ – Partitionof G. This increases the chromaticity of G and hence we get a contradiction. So for each v ∈ V \D there exists some u ∈ D such that u and v are not adjacent. Hence D is a total global dominating set of G.

Converse is obvious by theorem 3.14.

Theorem 3.18. Let G = (V, E) be a graph with diam(G) ≥ 5. Then γ_{tg}(G) ≤ $\boldsymbol{\gamma_{tstd}}$(G).

Proof: Let D be a $\boldsymbol{\gamma_{tstd}}$ - set of G. Then D is a total dominating set of G. Then by theorem 3.15, D is a total global dominating set of G. Hence γ_{tg}(G)≤ $\boldsymbol{\gamma_{tstd}}$(G).

4. Concluding remarks

The properties of dominating color transversal sets of graphs are studied by Manoharan in [1]. We have here explored some properties of total dominating color transversal sets of graphs. We are rigorously working on this topic to find many more exciting results.

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