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On Matrices Over Path Algebra

***Kanak Ray Chowdhury*[1]*, Md. Yasin Ali*[2]*, Abeda Sultana*[3]*, Nirmal Kanti Mitra*[4]**
and *A F M khodadad Khan*[5]

[1]Department of Mathematics, Mohammadpur Model School and College
Mohammadpur, Dhaka. **e-mail :** krchowdhury1975@yahoo.com
[2]School of Science and Engineering
University of Information Technology & Sciences, Dhaka
[3]Department of Mathematics, Jahangirnagar University, Savar, Bangladesh
[4]Mathematical and Physical Sciences
Bangladesh University of Business and Technology, Dhaka
[5]School of Engineering and Computer Science, Independent University
Bashundhara R/A, Dhaka, Bangladesh

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Abstract. In this paper, we consider a path algebra and discuss about matrices over path algebra. Some investigations on transitivity over path algebra are performed. Some properties and characterization for permanent of matrices over path algebra are established. Several inequalities over permanent are discussed. Also the adjoint matrix over path algebra are studied.

Keywords: Commutative, Idempotent, Transitive, Permanent, Incline, Adjoint.

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1. Introduction

Path algebras are useful in many areas such as design of switching circuits, automata theory, information system, dynamic programming and decision theory. For further examples [3, 8]. Boolean matrix, incline matrix [13] are the prototypical examples of matrices over path algebra. In 1999, Golan [4] worked on semirings and matrices over semirings. Transitive matrices are important type of matrices. Since the beginning of 1980s, many authors have studied this types for some special cases of path algebra e.g. [5]. In [3], Hasimoto considered transitivity of matrices over a general path algebra. A large number of work on permanent theory have been published [12, 13].

2. Preliminaries

In this section, we present some definitions and examples of algebraic structures of semiring. We support these definitions by some examples.

Definition 2.1. Let S be a non empty set with two binary operations **+** and **.** . Then the algebraic structure (S; +, .) is called ***a semiring*** iff $\forall a, b, c \in S$,

(i) (S; +) is monoid with identity element 0.
(ii) (S; +) is commutative.

(iii) (S; .) is monoid with identity element 1.
(iv) a. (b+c) = a.b + a.c and (b+c).a = b.a + c.a.
(v) 0. a = 0. a = 0 and $0 \neq 1$.

Example 2.1(a). (B = {0,1}, +, .) is a semiring, where + and . are defined by

+	0	1
0	0	1
1	1	1

.	0	1
0	0	0
1	0	1

Example 2.1(b). (I =[0,1]; +, .) is a semiring, where order in [0,1] is usual $\leq$ and + and . are defined as follows:

a + b = max{a,b}, a.b = min{a,b}.

Example 2.1(c). (L = {1, 2, 3, 4, 6, 8, 12, 24}; | , +, .) is a semiring, where

a + b = lcm{a , b}, a.b = gcd{a , b}.

Definition 2.2. Let (S; +, .) be a semiring**.** Then S is called ***commutative*** iff

$$\forall x, y \in S,\ x.y = y.x.$$

Example 2.2(a). $\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{Z}_0^+ = \{x \in \mathbb{Z} : x \geq 0\}$, $\mathbb{Q}_0^+ = \{x \in \mathbb{Q} : x \geq 0\}$ are commutative semirings .

Definition 2.3. Let (S; +, .) be a commutative semiring**.** It is called ***Boolean semiring*** iff $\forall x \in S$,

$$x.x = x.$$

Example 2.3(a).
(i) (B = {0,1}; +, .) is a *Boolean semiring*, where + and . are defined in Example 2.1(a).
(ii) (I =[0,1]; +, .) is a *Boolean semiring*, where order in [0,1] is usual $\leq$ and + and . are defined as follows:

a + b = max{a,b}, a.b = min{a,b}.

(iii) Let $X \neq \phi$ and P(X) is power set of X. + and . are defined by $A + B = A \cup B$ and A.B $= A \cap B; \forall A, B \in P(X)$. Then (P(X); +, .) is a Boolean semiring, where ϕ and X are zero and identity of P(X) respectively.

3. Transitive matrix over path algebra

In this section, we discuss path algebra, incline and some elementary properties of transitivity of matrices over path algebra.

Definition 3.1. Let (S; +, .) be a semiring**.** Then S is called ***path algebra*** iff

$$x + x = x\ ;\ \forall x \in S.$$

Example 3.1(a).
(1) (B={0, 1}; +, .) is a path algebra, where + and . are defined in Example 2.1(a).
(2) The fuzzy algebra $(F = [0.1]; \vee, T)$; where $\vee$ = max and T is a t-norm is a path algebra.

(3) Every bounded distributive lattice is a path algebra.

Definition 3.2. Let (S; +, .) be a path algebra. Then S is called ***commutative path algebra*** iff

$$x.y = y.x \; ; \; \forall x, y \in S.$$

Example 3.2(a).
(1) (B={0, 1}; +, .) is a commutative path algebra, where + and . are defined in Example 2.1(a).
(2) The fuzzy algebra $(F = [0.1]; \vee, T)$; where $\vee$ = max and T is a t-norm is a commutative path algebra.
(3) Every bounded distributive lattice is commutative path algebra.

Definition 3.3. Let (S; +, .) be a semiring**.** Then S is called an ***incline*** iff

$$x + 1 = 1 \; ; \; \forall x \in S.$$

Example 3.3(a). The fuzzy algebra $(F = [0.1]; \vee, T)$; where $\vee$ = max and T is a t-norm is an incline.

Proposition 3.4. Any incline is a path algebra.
Proof : Let (S; +, .) be an incline.
Then 1 + 1 = 1.
Now $\forall x \in S,$

$$x = x.1$$
$$= x.(1+1)$$
$$= x + x$$

So $x + x = x$; $\forall x \in S,$
Hence S is a path algebra. Δ

Remark 3.4(a). For any $a, b \in S; \; a \le b \Leftrightarrow a + b = b.$

Definition 3.5. Let (S; +, .) be a path algebra and $x \in S$. Then $x \in S$ is called ***transitive*** element iff

$$x^2 \le x\,.$$

Proposition 3.6. . Let (S; +, .) be an incline. Then every element in S is transitive.
Proof : Let (S; +, .) be an incline.
Then $x.+1 = 1$; $\forall x \in S,$
Now $x^2 + x = x\,(x+1)$

$$= x\,.1$$
$$= x$$

Therefore $x^2 \le x.$
Hence S is transitive. Δ

Definition 3.7. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$**.** Then A is said to be ***almost periodic*** if $\exists k, \; d \in Z^+$ such that $A^k = A^{k+d}$.

The least positive integer k is called ***the index*** and the least positive integer d is called ***the period*** of A. It is denoted by k(A) and d(A).

Definition 3.8. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. Then A is called ***transitive*** element iff $A^2 \le A$.

Definition 3.9. Let (S; +, .) be a path algebra and $A \in M_n(S)$. Then A is called ***invertible matrix*** iff $\exists G \in M_n(S)$ such that $AG = GA = I_n$, where I_n stands for identity matrix of order n.

Definition 3.10. Let (S ; +,.) be a path algebra and $A \in M_n(S)$. Then A is called ***a permutation matrix*** if every element of A is either 0 or 1 and each row and each column contains exactly one 1.

Example 3.10(a). $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is a permutation matrix.

Remark 3.10(b). A permutation matrix $A \in M_n(S)$ is clearly invertible over S and A^T is inverse of A.

From Example 3.10(a), $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$\therefore \quad A^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Clearly $A^T A = AA^T = I$

Therefore A^T is inverse of A.

Proposition 3.11. Let (S; +, .) be a path algebra and $A \in M_n(S)$. Then

(1) A is transitive iff $a_{ik}a_{kj} \le a_{ij}$ for all $i,\ j,\ k \in \{1, 2, 3, ..., n\}$

(2) A is transitive iff PAP^T is transitive for any $n \times n$ permutation matrix P.

Proof:

(1) Let A be transitive.

Then $A^2 \le A$

$\Rightarrow AA \le A$

$\Rightarrow \sum_{k=1}^{n} a_{ik}a_{kj} \le (a_{ij})$; where $i, j, k \in \{1,2,3,........,n\}$

$\Rightarrow (a_{ik})(a_{kj}) \le (a_{ij})$

Conversely suppose

$$(a_{ik})(a_{kj}) \le (a_{ij})$$

Now AA = $\sum_{k=1}^{n} a_{ik}a_{kj} \le (a_{ik})(a_{kj}) \le (a_{ij})$ = A

Hence $A^2 \le A$.

(2) Let A be transitive.

Then $A^2 \le A$

Now $(PAP^T)^2 = (PAP^T)(PAP^T)$

$$= (PA)(P^T P)(AP^T)$$
$$= (PA)I_n(AP^T)$$
$$= P(AI_nA)P^T$$
$$= P(AA)P^T$$
$$= PA^2P^T$$

Hence $(PAP^T)^2 \le PAP^T$

Conversely

$$(PAP^T)^2 \le PAP^T$$

Then $(PAP^T)(PAP^T) \le PAP^T$

$\Rightarrow (PA)(P^T P)(AP^T) \le PAP^T$

$\Rightarrow (PA)I_n(AP^T) \le PAP^T$

$\Rightarrow P(AI_nA)P^T \le PAP^T$

$\Rightarrow PA^2P^T \le PAP^T$

$\Rightarrow P^T PA^2P^T P \le P^T PAP^T P$

$\Rightarrow (P^T P)A^2(P^T P) \le (P^T P)A(P^T P)$

$\Rightarrow I_nA^2I_n \le I_nAI_n$

$\Rightarrow A^2 \le A$

Hence A is transitive. Δ

4. Permanent and adjoint of matrix over path algebra

In this section, we discuss permanent and adjoint of matrices over path algebra. Some properties also established.

Definition 4.1. Let Let (S; +, .) be a path algebra and $A \in M_n(S)$. Then the ***permanent of A*** denoted by *perA* is defined as:

$$perA = \sum_{\sigma \in S_n} a_{1\sigma(1)}\, a_{2\sigma(2)} a_{3\sigma(3)} \ldots\ldots\ldots a_{n\sigma(n)}$$

Where S_n denotes the symmetric group of the set $i \in \{1, 2, 3, \ldots\ldots\ldots, n\}$.

Remark 4.2. The partial relation $\leq$ over $M_n(S)$ is defined as:

$$A \leq B \Leftrightarrow a_{ij} \leq b_{ij} \;;\; \forall i,\; j$$
$$A \leq B \Leftrightarrow A + B = B.$$

Proposition 4.3. Let (S; +, .) be a commutative path algebra and $A, B \in M_n(S)$. Then $per(AB) \geq per(A).per(B).$

Proof: per (AB) = $\sum_{\sigma \in S_n} \left(\sum_{\xi=1}^{n} a_{1\xi} b_{1\sigma(1)} \sum_{\xi=1}^{n} a_{2\xi} b_{2\sigma(2)} \ldots\ldots\ldots\ldots\ldots \sum_{\xi=1}^{n} a_{n\xi} b_{n\sigma(n)}\right)$

$$= \sum_{\xi_1, \xi_2, \ldots\ldots\ldots\ldots \xi_n} \left(\sum_{\sigma \in S_n} a_{1\xi_1} a_{2\xi_2} a_{3\xi_3} \ldots\ldots a_{n\varsigma_n} b_{\varsigma_1\sigma(1)} b_{\varsigma_2\sigma(2)} b_{\varsigma_3\sigma(3)} \ldots\ldots\ldots b_{\varsigma_n\sigma(n)} \right)$$

$$\geq \sum_{\pi \in S_n} (a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} \ldots\ldots\ldots a_{n\pi(n)} \sum_{\sigma \in S_n} b_{\pi(1)\sigma(1)} b_{\pi(2)\sigma(2)} b_{\pi(3)\sigma(3)} \ldots\ldots\ldots b_{\pi(n)\sigma(n)})$$

$$= \sum_{\pi \in S_n} a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} \ldots\ldots\ldots a_{n\pi(n)}\; per(B)$$

$$= per(A).\, per(B)$$

Hence $per(AB) \geq per(A).per(B).$ Δ

Definition 4.4. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. Then A is called ***idempotent*** iff $A^2 = A.$

Proposition 4.5. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. If A is idempotent with $per(A) \geq 1,$ then $per(A)$ is idempotent.

Proof : Since A is idempotent, so $A^2 = A.$

By Proposition 4.3 we get

$$per(AB) \geq per(A).per(B). \qquad \text{....(i)}$$

Putting A for B in (i)

$$per(AA) \geq per(A).per(A)$$
$$\Rightarrow per(A^2) \geq (per(A))^2$$
$$\Rightarrow per(A) \geq (per(A))^2$$
$$\Rightarrow (per(A))^2 \leq per(A) \qquad \text{....(ii)}$$

We have

$$per(A) \geq 1$$
$$\Rightarrow per(A) per(A) \geq per(A)$$
$$\Rightarrow (per(A))^2 \geq per(A) \qquad \text{....(iii)}$$

From (ii) and (iii) we get

$$(per(A))^2 = per(A). \qquad \Delta$$

Definition 4.6. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. Then A is said to be ***nilpotent matrix*** if $\exists k \in Z^+$ such that $A^k = 0.$

Definition 4.7. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. Then A is called ***irreflexive*** element iff $a_{ii} = 0$; where $i \in \{1, 2, 3, ...,n\}$.

Proposition 4.8. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$.

(i) If $(A^k)_{ii} = 0$, for all $i, k \in \{1, 2, 3, ...,n\}$, then A is nilpotent.

(ii) If A is irreflexive and transitive, then A is nilpotent.

Proof : Trivial.

Definition 4.9. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$; $n \geq 2$. The matrix B is said to be ***adjoint matrix of matrix A*** if $b_{ij} = |A_{ji}|$; $1 \leq i, j \leq n$, where A_{ji} is matrix of order $n-1$ formed by delating row j and column i from A.
It is denoted by $adj(A)$.

Proposition 4.10. Let (S; +, .) be a commutative path algebra and $A, B \in M_n(S)$ and $B \leq A$. Then (1) $adjB \leq adjA$

(2) if A is nilpotent then B is nilpotent and $h(B) \leq h(A)$.

Proof: (1)

Let $A, B \in M_n(S)$ and $B \leq A$.

Then $(perB(i|j))^T_{n\times n} \leq (perA(i|j))^T_{n\times n}$

$$\Rightarrow adjB \leq adjA$$

(2) Let A be nilpotent.
So $\exists k \in Z^+$ such that $A^k = 0$.
Let

$$l < k.$$

Since $B \leq A$, so

$$B^l \leq A^k$$
$$\Rightarrow B^l \leq 0$$
$$\Rightarrow B^l = 0.$$

Hence B is nilpotent.
Again

$$B^l \leq A^k$$
$$\Rightarrow (B^l)_{ij} \leq (A^k)_{ij} \text{ ; where } i, j, k, l \in \{1, 2, 3, ...,n\}$$

From this we get

nilpotent index of B $\leq$ nilpotent index of A

$$\Rightarrow l(B) \leq k(A)$$

$\Rightarrow h(B) \le h(A)$. Δ

Proposition 4.11. Let (S; +, .) be a commutative path algebra and $A, B \in M_n(S)$. Then

(1) $adj(A) + adj(B) \le adj(A+B)$

(2) $(adj(A))^T = adj(A^T)$.

Proof:

(1) We have $A \le A+B$ and $B \le A+B$.

By Proposition 4.10 (1), we get

$$adj(A) \le adj(A+B) \quad \text{....(i)}$$

and $adj(B) \le adj(A+B)$ ...(ii)

From (i) and (ii) we get

$$adj(A) + adj(B) \le adj(A+B).$$

(2) It is obvious. Δ

Definition 4.12. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. Then A is called ***symmetric*** iff $A^T = A$.

Definition 4.13. Let (S; +, .) be a commutative path algebra and $A, B, C \in M_n(S)$. Then

$$A \circ B = C \text{ iff } c_{ij} = \prod_{k=1}^{n} (a_{ik} + b_{kj}) \text{ for any } i, j \in \{1, 2, 3, \ldots\ldots\ldots, n\}.$$

Proposition 4.14. Let (S; +, .) be a path algebra and $A, B, C, D \in M_n(S)$. Then

(i) $(B \circ C)^T = C^T \circ B^T$

(ii) If $A \le B$ then $D \circ A \le D \circ B$ and $A \circ C \le B \circ C$.

Proof : (i) $B \circ C = \prod_{k=1}^{n} (b_{ik} + c_{kj})$,

For i = 1,2, …..,m
j = 1, 2, …..,l.

$$(B \circ C)^T = (\prod_{k=1}^{n} (b_{ik} + c_{kj}))^T$$

$$= \prod_{k=1}^{n} (c_{jk} + b_{ki})$$

Now $C^T = c_{jk}$, $B^T = b_{ki}$

$$C^T \circ B^T = c_{jk} \circ b_{ki}$$

$$= \prod_{k=1}^{n} (c_{jk} + b_{ki})$$

Therefore $(B \circ C)^T = C^T \circ B^T$ Δ

(ii) Let

$$A \le B .$$

Then $a_{ij} \le b_{ij}$; for i = 1, 2,…..,m

j = 1, 2, …..,n

$$D \circ A = \prod_{i=1}^{m} (d_{ti} + a_{ij}) \le \prod_{i=1}^{m} (d_{ti} + b_{ij})$$

Therefore $D \circ A \le D \circ B$.

Again

$$A \circ C = \prod_{j=1}^{n} (a_{ij} + c_{jk}) \le \prod_{j=1}^{n} (b_{ij} + c_{jk})$$

Therefore $A \circ C \le B \circ C.$ Δ

Definition 4.15. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. Then A is called ***reflexive*** element iff $a_{ii} = 1$; where $i \in \{1, 2, 3, \ldots\ldots\ldots, n\}$.

Definition 4.16. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. Then A is called ***weekly reflexive*** element iff $a_{ii} \ge a_{ij}$; where $i, j \in \{1, 2, 3, \ldots\ldots\ldots, n\}$.

Definition 4.17. Let (S; +, .) be a commutative path algebra and $A \in M_n(S)$. Then A is called ***nearly irreflexive*** element iff $a_{ii} \le a_{ij}$; where $i, j \in \{1, 2, 3, \ldots\ldots\ldots, n\}$.

Proposition 4.18. Let (S; +, .) be a path algebra and $A \in M_n(S)$. If A is nearly irreflexive and symmetric, then

(1) $A \circ A \le A$

(2) $A \circ A$ is symmetric and nearly irreflexive.

(3) A^2 is weekly reflexive.

Proof : (1) Let

$$T = A \circ A$$

Then

$$t_{ij} = \prod_{k=1}^{n} (a_{ik} + a_{kj}) \quad \text{….(i)}$$

$$\le (a_{ii} + a_{ij})$$

$$\le a_{ij}$$

Hence $A \circ A \le A$.

(2) Now

$$t_{ji} = \prod_{k=1}^{n} (a_{jk} + a_{ki})$$

$$= \prod_{k=1}^{n} (a_{kj} + a_{ik}) \qquad [\because \text{ A is Symmetric}]$$

$$= \prod_{k=1}^{n} (a_{ik} + a_{kj})$$

$$= t_{ij}$$

Hence T is symmetric.
Again

$$t_{ii} = \prod_{k=1}^{n} (a_{ik} + a_{ki})$$

$$= \prod_{k=1}^{n} (a_{ik} + a_{ik})$$

$$= \prod_{k=1}^{n} a_{ik}$$

$$\leq \prod_{k=1}^{n} (a_{ik} + a_{kj})$$

$$= t_{ij}$$

Hence $A \circ A$ is nearly irreflexive.

(3) Let $S = A^2$**.**

Then $s_{ii} = \sum_{k=1}^{n} a_{ik} a_{ki}$

$$= \sum_{k=1}^{n} a_{ik} a_{ik}$$

$$= \sum_{k=1}^{n} a_{ik}$$

$$\geq \sum_{k=1}^{n} a_{ik} a_{kj}$$

$$\geq s_{ij}$$

Hence A^2 is weekly reflexive. Δ

Proposition 4.19. Let (S; +, .) be a path algebra and $A \in M_n(S)$. If A is nearly irreflexive and symmetric, then $A \circ A^T$ is symmetric and nearly irreflexive.

Proof : Let $S = A \circ A^T$.

Then $s_{ii} = \prod_{k=1}^{n} (a_{ik} + a_{ik})$

$$= \prod_{k=1}^{n} a_{ik} \quad \leq \prod_{k=1}^{n} (a_{ik} + a_{jk}) = s_{ij}$$

Hence $A \circ A^T$ is nearly irreflexive.

By **Proposition 4.14**, $(A \circ A^T)^T = (A^T)^T \circ A^T = A \circ A^T$. Δ

Proposition 4.20. Let (S; +, .) be a path algebra and $A \in M_n(S)$. If A is irreflexive and transitive, then (1) $A \circ A^T = 0$

(2) $A^T \circ A = 0$

Proof : Let $S = A \circ A^T$

Then $s_{ij} = \prod_{k=1}^{n} (a_{ik} + a_{jk}) \leq (a_{ij} + a_{ji})(a_{ij} + a_{jj}) = a_{ij}a_{ji} \leq a_{ii} = 0$

Hence $A \circ A^T = 0$.

The proof of (2) is similar.

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