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# Lebesgue Measure of Generalized Cantor Set

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***Abstract.*** In this paper, we discuss the construction and properties of generalized Cantor set. We show that this special type of set is measurable set, Borel set as well as Borel measurable whose Lebesgue measure is zero. We also prove several interesting lemma, theorems, and propositions relating to generalized Cantor set.

***Keywords:*** Cantor set, Measurable set, Borel set, Borel measure and Lebesgue measure

***AMS Mathematics Subject Classification (2010):*** 28A20, 28A80

## 1. Introduction

A Fractal, is defined by Mandelbrot as "is a shape made of parts similar to the whole in some way" [1]. Fractal is a geometric object that possesses the two properties: self-similar and non-integer dimensions. So a fractal is an object or quantity which displays self-similarity. The Cantor set is the prototypical fractal [2]. The Cantor sets were discovered by the German Mathematician George Cantor in the late 19th to early 20th centuries (1845-1918) [3]. He introduced fractal which has come to be known as the Cantor set, or Cantor dust.

We studied Cantor set and found generalized Cantor set and proved its dynamical behaviors and fractal dimensions [4]. The Cantor middle $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \cdots$ sets, in general, Cantor middle $\{\frac{1}{2m-1} : 2 \le m < \infty\}$ is called generalized Cantor set and it is denoted by $C_{1/(2m-1)}$. Since the Cantor set is the prototypical fractal, we would like to study the generalized Cantor set in measure space, which is defined by an algorithm and also defined by the shrinking process [9].

A measure is a countably additive, non-negative, extended real-valued function defined on a $\sigma$-algebra. There are different types of measure such as Borel measure, Lebesgue measure, Probability measure, Counting measure and Random measure. Let $X$ be a locally compact Hausdorff space, let $\Sigma$ be the smallest $\sigma$-algebra that contains the open sets (or, equivalently, the closed sets) of $X$; this is known as the $\sigma$-algebra of Borel sets. Any measure $\mu$ defined on the $\sigma$-algebra of Borel sets is called a Borel measure. Every Borel measure on $[0,1]$ ($\mathbf{R}$ or $\mathbf{R}^n$ etc.) possesses a unique completion which is a Lebesgue measure. If $\mu$ is both inner regular and locally finite, it is called a

Random measure [12]. In this paper, we discuss the construction and some properties of generalized Cantor set. We show that this special type of set is measurable set, Borel set as well as Borel measurable whose Lebesgue measure is zero. We also prove several interesting lemma, theorems, and propositions relating to generalized Cantor set.

## 2. Preliminaries

**Definition 2.1.** A non empty set $\Gamma \subset \mathbf{R}$ is called a Cantor set if
(a) $\Gamma$ is closed and bounded. (b) $\Gamma$ contains no intervals.
(c) Every point in $\Gamma$ is an accumulation point of $\Gamma$.

**Definition 2.2.** The outer measure of any interval $I$ on $\mathbf{R}$ with endpoints $a < b$ is $b - a$ and is denoted as $\lambda^*(I) = b - a$. A set $E \subset \mathbf{R}$ is said to be outer measure (or measurable) if, for all $A \subset \mathbf{R}$ one has $\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap \overline{E})$.

**Definition 2.3.** The inner measure of any set $A \subset E$, denoted $\lambda_*(A)$, is defined as $\lambda_*(A) = \lambda^*(E) - \lambda^*(E \setminus A)$, where $E \setminus A$ is the complement of $A$ with respect to $E$.

**Definition 2.4.** [7] If $E$ is a measurable set, we define the Lebesgue measure $\lambda(E)$ to be the outer measure of $E$. That is, $\lambda(E) = \lambda^*(E)$.

**Definition 2.5.** [8] A set $A \subset E$ is Lebesgue measurable or measurable if $\lambda^*(A) = \lambda_*(A)$, in which case the measure of $A$ is denoted simply by $\lambda(A)$ and is given by $\lambda(A) = \lambda^*(A) = \lambda_*(A)$.

**Definition 2.6.** A Borel set is any set in topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection and relative complement. Borel sets are named after Emile Borel.

**Definition 2.7.** A collection $\mathcal{B}$ of subsets of a set $X$ is called a $\sigma$-algebra if $\mathcal{B}$ satisfies the following axioms:
A1: $X \in \mathcal{B}$, A2: $A \in \mathcal{B} \Rightarrow X \setminus A \in \mathcal{B}$,
A3: If $(A_n)_{n \in \mathbf{N}}$ is sequence in $\mathcal{B}$ then $\bigcup_n A_n \in \mathcal{B}$.

**Definition 2.8.** The Borel $\sigma$-algebra of a set $X$ is the smallest $\sigma$-algebra of $X$ that contains all of the open balls in $X$. Any element of a Borel $\sigma$-algebra is a Borel set.

**Definition 2.9.** Let $X$ be a set and $\tau$ be a collection of subsets of $X$. Then $(X, \tau)$ be a topological space [10]. The Borel or topological $\sigma$-algebra $B(\tau)$ of a topological space $(X, \tau)$ is the $\sigma$-algebra generated by $\tau$.

**Theorem 2.10.** Every Borel set is measurable. In particular each open set and each closed set is measurable.
**Proof:** The proof can be found in Real Analysis [5].

**Definition 2.11.** Let $X=[a,b]$ be a closed set and let $\mathcal{B}$ be a collection of subsets of $X$. A set function $\mu$ on $\mathcal{B}$ (i.e. $\mu:\mathcal{B}\to[0,\infty]$ is called a measure if the following properties hold:

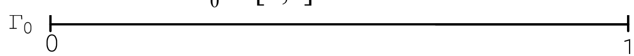
1. $0\le\mu(A)\le b-a$ for all $A\in\mathcal{B}$  2. $\mu(\phi)=0$
3. $\mu(A)\le\mu(B)$ for all $A,B\in\mathcal{B},\ A\subset B$
4. If $A_1,A_2,A_3,\ldots$ are in $\mathcal{B}$, with $A_i\cap A_j=\phi$ for $i\ne j$,

then $\mu(A)=\mu\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}\mu(A_i)$. The pair $(X,\mathcal{B})$ is called a measurable space and the triple $(X,\mathcal{B},\ \mu)$ is a measure space.

## 3. Generalized Cantor set (Cantor middle $\frac{1}{2m-1}$, $(2\le m<\infty)$ set)

### 3.1.1. Construction of the Cantor middle $\frac{1}{3}$ set

We start with the closed interval $\Gamma_0=[0,1]$.




Remove the middle open third. This leaves a new set $\Gamma_1=[0,\frac{1}{3}]\cup[\frac{2}{3},1]$.


Each iteration through the algorithm removes the open middle third from each segment of the previous iteration. Thus the next set would be

$$\Gamma_2=[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1].$$

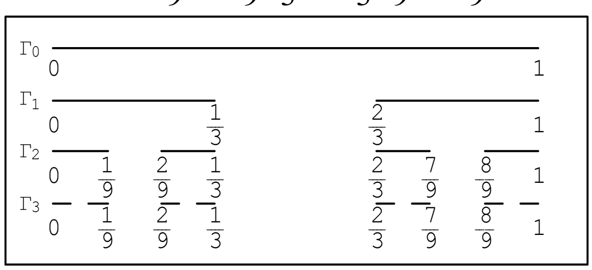



**Figure 3.1.** Construction of the Cantor middle $\frac{1}{3}$ set

In general, after $n$ times iterations, we obtain $\Gamma_n$ which as follows

$$\Gamma_n=[0,\frac{1}{3^n}]\cup[\frac{2}{3^n},\frac{3}{3^n}]\cup\cdots\cup[\frac{3^n-3}{3^n},\frac{3^n-2}{3^n}]\cup[\frac{3^n-1}{3^n},1],\ \text{where } n\ge1.$$

Therefore we construct a decreasing sequence $(\Gamma_n)$ of closed sets, that is $\Gamma_{n+1}\subseteq\Gamma_n$ for all $n\in\mathbf{N}$, so that every $\Gamma_n$ consists of $2^n$ closed intervals all of which the same leng

$\frac{1}{3^n}$. The Cantor ternary set, which we denote $C_{1/3}$, is the "limiting set" of this process, that is, $C_{1/3} = \bigcap_{n=1}^{\infty} \Gamma_n$ [7] and call it the Cantor middle $\frac{1}{3}$ set.

Alternative process of constructing $C_{1/3}$ is in physical terms as taking a length of string and repeatedly cutting it into shorter pieces. If we think first piece as the interval $[0,1]$ and cut it at the points 1/2, then it becomes two pieces of string each with two endpoints such as the intervals $[0,1/2]$, and $[1/2,1]$. In order to make all these pieces disjoint subsets of $\mathbf{R}$ one can image the string as being stretched so tightly that each time it is cut, it pulls apart at the cut and shrinks to $2/3$ of its length, so after the first cut, $[0,1/2]$ shrinks to $[0,1/3]$, $[1/2,1]$ shrinks to $[2/3,1]$. Then at the next stage we cut $[0,1/3]$ at the point 1/6, and then two pieces are $[0,1/6]$, $[1/6,1/3]$, shrink to $[0,1/9]$ and $[2/9,1/3]$. similarly for the piece $[2/3,1]$, and so on.

### 3.1.2. Properties of the Cantor middle $\frac{1}{3}$ set

#### 3.1.2.1. The set $C_{1/3}$ is disconnected

The set $C_{1/3}$ is totally disconnected since it was constructed so as to contain no intervals other than points. Namely, if $C_{1/3}$ contained an interval of positive length $\varepsilon$ then this interval would be contained in each $\Gamma_n$, but $\Gamma_n$ contains no interval of length greater than $\frac{1}{3^n}$ so if *n* is chosen to be large enough so that $\frac{1}{3^n}$ is less than $\varepsilon$, then there is no interval of length $\varepsilon$ in $\Gamma_n$.

#### 3.1.2.2 [11] The set $C_{1/3}$ contains no intervals

We will show that the length of the complement of the set $C_{1/3}$ is equal to $1$, hence $C_{1/3}$ contains no intervals. At the $n^{th}$ stage, we are removing $2^{n-1}$ intervals from the previous set of intervals, and each one has length of $\frac{1}{3^n}$. The length of the removing intervals within $[0,1]$ after an infinite number of removals is

$$\sum_{n=1}^{\infty} 2^{n-1}(\frac{1}{3^n}) = \frac{1}{3}\sum_{n=1}^{\infty}(\frac{2}{3})^{n-1} = \frac{1}{3}\sum_{n=0}^{\infty}(\frac{2}{3})^{n} = \frac{1/3}{1-2/3} = 1$$

Thus, we are removing a length of $1$ from the unit interval $[0,1]$ which has a length of 1.

**Alternative method:**

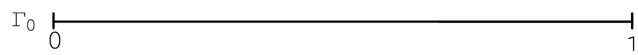
Note that in the first iteration we removed 1/3, in the second iteration we removed 2/9, in the third iteration we removed 4/27, and in the fourth iteration we removed 8/81, and so forth. This is a geometric series with first term $a=\frac{1}{3}$ and common ratio $r=\frac{2}{3}$. This converges, and the sum is $S_\infty=\frac{1/3}{1-2/3}=1$.

Thus the length of the complement of the set $C_{1/3}$ is equal to $1$.

Therefore, the total length of $C_{1/3}$ is 0, which means it has no intervals.

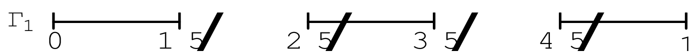
**3.2.1. Construction of the Cantor middle $\frac{1}{5}$ set**

We start with the closed interval $\Gamma_0=[0,1]$.




Remove the middle open interval $(1/5,2/5)$ and $(3/5,4/5)$. This leaves a new set

$$\Gamma_1=[0,\frac{1}{5}]\cup[\frac{2}{5},\frac{3}{5}]\cup[\frac{4}{5},1].$$




Each iteration through the algorithm removes the open 2nd and 4th interval from each segment of the previous iteration. Thus the next set would be

$$\Gamma_2=[0,\frac{1}{25}]\cup[\frac{2}{25},\frac{3}{25}]\cup[\frac{4}{25},\frac{1}{5}]\cup[\frac{2}{5},\frac{11}{25}]\cup[\frac{12}{25},\frac{13}{25}]\cup$$
$$[\frac{14}{25},\frac{3}{5}]\cup[\frac{4}{5},\frac{21}{25}]\cup[\frac{22}{25},\frac{23}{25}]\cup[\frac{24}{27},1].$$

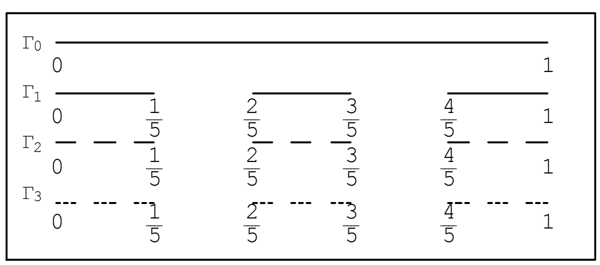



**Figure 3.2:** Construction of the Cantor middle $\frac{1}{5}$ set

In general, after $n$ times iterations, we obtain $\Gamma_n$ which as follows

$$\Gamma_n=[0,\frac{1}{5^n}]\cup[\frac{2}{5^n},\frac{3}{5^n}]\cup\cdots\cup[\frac{5^n-3}{5^n},\frac{5^n-2}{5^n}]\cup[\frac{5^n-1}{5^n},1],\ \text{where}\ n\geq 1.$$

Therefore, we construct a decreasing sequence $(\Gamma_n)$ of closed sets, that is, $\Gamma_{n+1}\subseteq\Gamma_n$ for all $n\in\mathbf{N}$, so that every $\Gamma_n$ consists of $3^n$ closed intervals all of which the same length $\frac{1}{5^n}$. We define $C_{1/5}=\bigcap_{n=1}^{\infty}\Gamma_n$ and call it the Cantor middle $\frac{1}{5}$ set.

Alternative process of constructing $C_{1/5}$ is in physical terms as taking a length of string and repeatedly cutting it into shorter pieces. If we think first piece as the interval $[0,1]$ and cut it at the points 1/3 and 2/3, then it becomes three pieces of string each with two endpoints such as the intervals $[0,1/3]$, $[1/3,2/3]$, and $[2/3,1]$. In order to make all these pieces disjoint subsets of $\mathbf{R}$ one can image the string as being stretched so tightly that each time it is cut, it pulls apart at the cut and shrinks to $3/5$ of its length, so after the first cut, $[0,1/3]$ shrinks to $[0,1/5]$, $[1/3,2/3]$ shrinks to $[2/5,3/5]$, and $[2/3,1]$ shrinks to $[4/5,1]$. Then at the next stage we cut $[0,1/5]$ at the points 1/15 and 2/15 and the three pieces $[0,1/15]$, $[1/15,2/15]$, and $[2/15,1/5]$ shrink to $[0,1/25]$, $[2/25,3/25]$, and $[4/25,1/5]$, similarly for the pieces $[2/5,3/5]$, and $[4/5,1]$, and so on.

### 3.2.2. Properties of the Cantor middle $\frac{1}{5}$ set

#### 3.2.2.1. The set $C_{1/5}$ is disconnected

The set $C_{1/5}$ is totally disconnected since it was constructed so as to contain no intervals other than points. Namely, if $C_{1/5}$ contained an interval of positive length $\varepsilon$ then this interval would be contained in each $\Gamma_n$, but $\Gamma_n$ contains no interval of length greater than $\frac{1}{5^n}$ so if $n$ is chosen to be large enough so that $\frac{1}{5^n}$ is less than $\varepsilon$, then there is no interval of length $\varepsilon$ in $\Gamma_n$.

#### 3.2.2.2. The set $C_{1/5}$ contains no intervals

We will show that the length of the complement of the set $C_{1/5}$ is equal to 1, hence $C_{1/5}$ contains no intervals. At the $n^{th}$ stage, we are removing $2.3^{n-1}$ intervals from the previous set of intervals, and each one has length of $\frac{1}{5^n}$. The length of the removing intervals within $[0,1]$ after an infinite number of removals is

$$\sum_{n=1}^{\infty} 2.3^{n-1}(\frac{1}{5^n}) = \frac{2}{5}\sum_{n=1}^{\infty}(\frac{3}{5})^{n-1} = \frac{2}{5}\sum_{n=0}^{\infty}(\frac{3}{5})^{n} = \frac{2/5}{1-3/5} = 1$$

Thus, we are removing a length of 1 from the unit interval $[0,1]$ which has a length of 1.

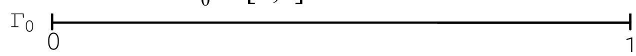
**Alternative method:**
Note that in the first iteration we removed 2/5, in the second iteration we removed 6/25, in the third iteration we removed 18/125, and so forth. This is a geometric series with first term $a = \frac{2}{5}$ and common ratio $r = \frac{3}{5}$. This converges, and the sum is $S_\infty = \frac{2/5}{1-3/5} = 1$.

Thus the length of the complement of the set $C_{1/5}$ is equal to 1.

Therefore, the total length of $C_{1/5}$ is 0, which means it has no intervals.

### 3.3.1. Construction of the Cantor middle $\frac{1}{7}$ set

We start with the closed interval $\Gamma_0 = [0,1]$.




Remove the middle open interval $(1/7, 2/7)$, $(3/7, 4/7)$, and $(5/7, 6/7)$.

This leaves a new set $\Gamma_1 = [0, \frac{1}{7}] \cup [\frac{2}{7}, \frac{3}{7}] \cup [\frac{4}{7}, \frac{5}{7}] \cup [\frac{6}{7}, 1]$.


Each iteration through the algorithm removes the open 2nd, 4th, and 6th interval from each segment of the previous iteration. Thus the next set would be

$$\Gamma_2 = [0, \frac{1}{49}] \cup [\frac{2}{49}, \frac{3}{49}] \cup [\frac{4}{49}, \frac{5}{49}] \cup [\frac{6}{49}, \frac{7}{49}] \cup \cdots \cup [\frac{46}{49}, \frac{47}{49}] \cup [\frac{48}{49}, 1].$$

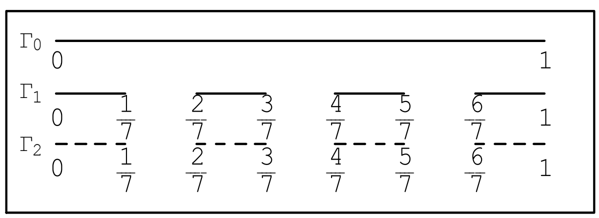



**Figure 3.3:** Construction of the Cantor middle $\frac{1}{7}$ set

In general, after $n$ times iterations, we obtain $\Gamma_n$ which as follows

$$\Gamma_n = [0, \frac{1}{7^n}] \cup [\frac{2}{7^n}, \frac{3}{7^n}] \cup \cdots \cup [\frac{7^n - 3}{7^n}, \frac{7^n - 2}{7^n}] \cup [\frac{7^n - 1}{7^n}, 1], \text{ where } n \ge 1.$$

Therefore we construct a decreasing sequence $(\Gamma_n)$ of closed sets, that is, $\Gamma_{n+1} \subseteq \Gamma_n$ for all $n \in \mathbf{N}$, so that every $\Gamma_n$ consists of $4^n$ closed intervals all of which the same length $\frac{1}{7^n}$. We define $C_{1/7} = \bigcap_{n=1}^{\infty} \Gamma_n$ and call it the Cantor middle $\frac{1}{7}$ set.

### 3.3.2. Properties of the Cantor middle $\frac{1}{7}$ set

#### 3.3.2.1. The set $C_{1/7}$ is disconnected

The set $C_{1/7}$ is totally disconnected since it was constructed so as to contain no intervals other than points. Namely, if $C_{1/7}$ contained an interval of positive length $\varepsilon$ then this interval would be contained in each $\Gamma_n$, but $\Gamma_n$ contains no interval of length greater than $\frac{1}{7^n}$ so if $n$ is chosen to be large enough so that $\frac{1}{7^n}$ is less than $\varepsilon$, then there is no interval of length $\varepsilon$ in $\Gamma_n$.

### 3.3.2.2. The set $C_{1/7}$ contains no intervals

We will show that the length of the complement of the set $C_{1/7}$ is equal to $1$, hence $C_{1/7}$ contains no intervals. At the $n^{th}$ stage, we are removing $3.4^{n-1}$ intervals from the previous set of intervals, and each one has length of $\frac{1}{7^n}$.

The length of the removing intervals within $[0,1]$ after an infinite number of removals is

$$\sum_{n=1}^{\infty} 3.4^{n-1}(\frac{1}{7^n}) = \frac{3}{7}\sum_{n=1}^{\infty}(\frac{4}{7})^{n-1} = \frac{3}{7}\sum_{n=0}^{\infty}(\frac{4}{7})^{n} = \frac{3/7}{1-4/7} = 1$$

Thus, we are removing a length of $1$ from the unit interval $[0,1]$ which has a length of $1$.

**Alternative method:**
Note that in the first iteration we removed 3/7, in the second iteration we removed 12/49, in the third iteration we removed 48/343 and so forth. This is a geometric series with first term $a = \frac{3}{7}$ and common ratio $r = \frac{4}{7}$. This converges, and the sum is $S_\infty = \frac{3/7}{1-4/7} = 1$.

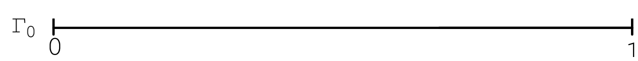
Thus the length of the complement of the set $C_{1/7}$ is equal to $1$.

Therefore, the total length of $C_{1/7}$ is 0, which means it has no intervals.

Similarly, we can construct and show the properties of the Cantor middle $\frac{1}{2m-1}, (2 \le m < \infty)$ set which is denoted by $C_{1/(2m-1)}$ and is called generalized Cantor set.

### 3.4.1. Construction of the Cantor middle $\frac{1}{2m-1}, (2 \le m < \infty)$ set

We start with the closed interval $\Gamma_0 = [0,1]$.




Remove the middle open interval

$$\left(\frac{1}{2m-1}, \frac{2}{2m-1}\right), \left(\frac{3}{2m-1}, \frac{4}{2m-1}\right), \dots, \left(\frac{2m-3}{2m-1}, \frac{2m-2}{2m-1}\right),$$

where $2 \le m < \infty$. This leaves a new set $\Gamma_1$ which will depend on the value of $m$.

In general, after $n$ times iterations, we obtain $\Gamma_n$ which as follows:

$$\Gamma_n = [0, \frac{1}{(2m-1)^n}] \cup [\frac{2}{(2m-1)^n}, \frac{3}{(2m-1)^n}] \cup \cdots$$
$$\cup [\frac{(2m-1)^n - 3}{(2m-1)^n}, \frac{(2m-1)^n - 2}{(2m-1)^n}] \cup [\frac{(2m-1)^n - 1}{(2m-1)^n}, 1],$$

Therefore, we construct a decreasing sequence $(\Gamma_n)$ of closed sets, that is $\Gamma_{n+1} \subseteq \Gamma_n$ for all $n \in \mathbf{N}$, so that every $\Gamma_n$ consists of $m^n$ closed intervals all of which the same length

$\frac{1}{(2m-1)^n}$. We define $C_{1/(2m-1)} = \bigcap_{n=1}^{\infty} \Gamma_n$ and call it the Cantor middle $\frac{1}{2m-1}$ set, where $2 \le m < \infty$ or generalized Cantor set.

**3.4.2. Properties of the Cantor middle $\frac{1}{2m-1}$ set**

**3.4.2.1 The set $C_{1/(2m-1)}$ is disconnected**

The set $C_{1/(2m-1)}$ is totally disconnected since it was constructed so as to contain no intervals other than points. Namely, if $C_{1/(2m-1)}$ contained an interval of positive length $\varepsilon$ then this interval would be contained in each $\Gamma_n$, but $\Gamma_n$ contains no interval of length greater than $\frac{1}{(2m-1)^n}$ so if *n* is chosen to be large enough so that $\frac{1}{(2m-1)^n}$ is less than $\varepsilon$, then there is no interval of length $\varepsilon$ in $\Gamma_n$.

**3.4.2.2. The set $C_{1/(2m-1)}$ contains no intervals**

We will show that the length of the complement of the set $C_{1/(2m-1)}$ is equal to 1, hence $C_{1/(2m-1)}$ contains no intervals. At the $n^{th}$ stage, we are removing $(m-1).m^{n-1}$ intervals from the previous set of intervals, and each one has length of $\frac{1}{(2m-1)^n}$. The length of the removing intervals within $[0,1]$ after an infinite number of removals is

$$\sum_{n=1}^{\infty}(m-1).m^{n-1}\left(\frac{1}{(2m-1)^n}\right) = \frac{m-1}{(2m-1)}\sum_{n=1}^{\infty}\left(\frac{m}{2m-1}\right)^{n-1} = \frac{m-1}{(2m-1)}\sum_{n=0}^{\infty}\left(\frac{m}{2m-1}\right)^{n} = 1$$

Thus, we are removing a length of 1 from the unit interval $[0,1]$ which has a length of 1.

**Alternative method:**

Note that in the first iteration we removed $(m-1)/(2m-1)$, in the second iteration we removed $m(m-1)/(2m-1)^2$, in the third iteration we removed $m^2(m-1)/(2m-1)^3$, and so on. This is a geometric series with first term $a = \frac{m-1}{2m-1}$ and common ratio $r = \frac{m}{2m-1}$. This converges, and the sum is $S_\infty = \frac{(m-1)/(2m-1)}{1-m/(2m-1)} = 1$. Therefore, the total length of $C_{1/(2m-1)}$ is 0, which means it has no intervals.

**3.4.2.3. The set $C_{1/(2m-1)}$ is nowhere dense**

A set S is said to be nowhere dense if the interior of the closure of S is empty. The closure of the set is the union of the set with the set of limit points. Since every point in the set $C_{1/(2m-1)}$ is a limit point of the set, the closure of the set is simply the set itself.

The interior of the set $C_{1/(2m-1)}$ must be empty, since no two points in the set are adjacent to each other. Thus the set $C_{1/(2m-1)}$ is nowhere dense.

**4. Lebesgue measure of generalized Cantor set**

**Lemma 4.1.** Let $X=[0,1]$ be a closed set and $\tau$ be a collection of subsets of $X$. Then $(X,\tau)$ be a topological space. Let $C_{1/(2m-1)}=\bigcap_{n\in\mathbf{N}}\Gamma_n$ be closed subsets in $X$. Then $C_{1/(2m-1)},(2\le m<\infty)$ is a Borel set as well as measurable set.

**Proof:** Since every intersection of closed sets is again closed set, $\bigcap_{n\in\mathbf{N}}\Gamma_n$ is closed set. By the definition of Borel set, $\bigcap_{n\in\mathbf{N}}\Gamma_n$ is a Borel set. Thus $C_{1/(2m-1)},(2\le m<\infty)$ is a Borel set. Since every Borel set is measurable, then $C_{1/(2m-1)},(2\le m<\infty)$ is measurable set. Thus $C_{1/(2m-1)},(2\le m<\infty)$ is a Borel set as well as measurable set.

**Theorem 4.2.** Let $X=[0,1]$ be a closed set and $\Sigma$ be a $\sigma$-algebra on $X$. Then $(X,\Sigma)$ is a measurable space. Let $(\Gamma_n)_{n\in\mathbf{N}}\in\Sigma$ be a collection of measurable sets. Then show that $C_{1/(2m-1)}\in\Sigma,$ where $2\le m<\infty.$

**Proof:** We know $C_{1/(2m-1)}=\bigcap_{n\in\mathbf{N}}\Gamma_n,\ (2\le m<\infty).$ For each $n\in\mathbf{N},\ \Gamma_n\in\Sigma.$

This implies that $X\setminus\Gamma_n\in\Sigma,$ by Axiom (A2) for $\sigma$-algebra.

Then $\bigcup_{n\in\mathbf{N}}(X\setminus\Gamma_n)\in\Sigma,$ by Axiom (A3) for $\sigma$-algebra.

This implies that $X\setminus\left(\bigcup_{n\in\mathbf{N}}(X\setminus\Gamma_n)\right)\in\Sigma,$ by Axiom (A2) for $\sigma$-algebra.

Now using De Morgan's laws, we have $\bigcup_{n\in\mathbf{N}}(X\setminus\Gamma_n)=X\setminus\bigcap_{n\in\mathbf{N}}\Gamma_n\in\Sigma$

and $X\setminus\left(\bigcup_{n\in\mathbf{N}}(X\setminus\Gamma_n)\right)=X\setminus\left(X\setminus\bigcap_{n\in\mathbf{N}}\Gamma_n\right)=\bigcap_{n\in\mathbf{N}}\Gamma_n\in\Sigma.$

Thus $C_{1/(2m-1)}\in\Sigma,$ where $2\le m<\infty.$

**Theorem 4.3.** Let $X=[0,1]$ be a closed set and $(X,\tau)$ be a topological space. Let $B(\tau)$ be the associated Borel $\sigma$-algebra. Let $(\Gamma_n)_{n\in\mathbf{N}}$ be closed subset in $X$. Then show that $C_{1/(2m-1)},(2\le m<\infty)$ is $B(\tau)$-measurable.

**Proof:** We know $C_{1/(2m-1)}=\bigcap_{n\in\mathbf{N}}\Gamma_n,\ (2\le m<\infty).$ Since $(\Gamma_n)_{n\in\mathbf{N}}$ is closed set in $X$, $C_{1/(2m-1)}$ is closed set in $X$. Then $X\setminus C_{1/(2m-1)}$ is open set.

By the definition of Borel $\sigma$-algebra, $X\setminus C_{1/(2m-1)}\in B(\tau).$

This implies that $X\setminus(X\setminus C_{1/(2m-1)})=C_{1/(2m-1)}\in B(\tau),$ by Axioms (A2) for $\sigma$-algebra.

Thus $C_{1/(2m-1)}, (2 \le m < \infty)$ is $B(\tau)$-measurable.

**Theorem 4.4.** If $\lambda(C_{1/(2m-1)}) = \lim_{n\to\infty} \lambda(\Gamma_n) = 0$, then $C_{1/(2m-1)}, (2 \le m < \infty)$ has Lebesgue measure zero.

**Proof:** We know $C_{1/(2m-1)} = \bigcap_{n\in\mathbf{N}} \Gamma_n,\ (2 \le m < \infty),$ where

$$\Gamma_n = [0, \frac{1}{(2m-1)^n}] \cup [\frac{2}{(2m-1)^n}, \frac{3}{(2m-1)^n}] \cup \cdots \cup [\frac{(2m-1)^n - 1}{(2m-1)^n}, 1],\ (n \ge 1).$$

Now $\lambda(C_{1/(2m-1)}) = \lim_{n\to\infty} \lambda(\Gamma_n)$

$$= \lim_{n\to\infty} \lambda([0, \frac{1}{(2m-1)^n}] \cup [\frac{2}{(2m-1)^n}, \frac{3}{(2m-1)^n}] \cup \cdots$$

$$\cup [\frac{(2m-1)^n - 3}{(2m-1)^n}, \frac{(2m-1)^n - 2}{(2m-1)^n}] \cup [\frac{(2m-1)^n - 1}{(2m-1)^n}, 1])$$

$$= \lim_{n\to\infty} [\lambda([0, \frac{1}{(2m-1)^n}]) + \lambda([\frac{2}{(2m-1)^n}, \frac{3}{(2m-1)^n}]) + \cdots$$

$$+ \lambda([\frac{(2m-1)^n - 3}{(2m-1)^n}, \frac{(2m-1)^n - 2}{(2m-1)^n}]) + \lambda([\frac{(2m-1)^n - 1}{(2m-1)^n}, 1])]$$

$$= \lim_{n\to\infty} [\frac{1}{(2m-1)^n} + \frac{3}{(2m-1)^n} - \frac{2}{(2m-1)^n} + \cdots$$

$$+ \frac{(2m-1)^n - 2}{(2m-1)^n} - \frac{(2m-1)^n - 3}{(2m-1)^n} + 1 - \frac{(2m-1)^n - 1}{(2m-1)^n}] = 0$$

Therefore, $\lambda(c_{1/2m-1}) = 0$. Hence $C_{1/2m-1}$, $(2 \le m < \infty)$ has Lebesgue measure zero.

**Alternative method:**

**Theorem 4.5.** The generalized Cantor set $C_{1/(2m-1)}, (2 \le m < \infty)$ is measurable and has Lebesgue measure zero.

**Proof:** We know $C_{1/(2m-1)} = \bigcap_{n\in\mathbf{N}} \Gamma_n,\ (2 \le m < \infty),$ where

$$\Gamma_n = [0, \frac{1}{(2m-1)^n}] \cup [\frac{2}{(2m-1)^n}, \frac{3}{(2m-1)^n}] \cup \cdots \cup [\frac{(2m-1)^n - 1}{(2m-1)^n}, 1],\ (n \ge 1).$$

By Lemma 4.1, $C_{1/(2m-1)}$ is a Borel set as well as measurable set. From the construction of $C_{1/(2m-1)}, (2 \le m < \infty)$, we remove $(m-1).m^{n-1}$ disjoint intervals from each previous segments and each having length $1/(2m-1)^n$, where $n \ge 1$.

Thus we will remove a total length

$$\sum_{n=1}^{\infty}(m-1).m^{n-1}.\frac{1}{(2m-1)^n}=\frac{m-1}{2m-1}\sum_{n=1}^{\infty}(m/(2m-1))^{n-1}$$
$$=\frac{m-1}{2m-1}\sum_{n=0}^{\infty}(m/(2m-1))^n=\frac{m-1}{2m-1}\left(\frac{1}{1-m/(2m-1)}\right)=1.$$

Therefore, $C_{1/(2m-1)}$ is obtained by removing a total length 1 from the unit interval $[0,1]$. Thus $\lambda(I\setminus C_{1/(2m-1)})=1$. Since $\lambda(I)=\lambda(C_{1/(2m-1)})+\lambda(I\setminus C_{1/(2m-1)})$, then $\lambda(C_{1/(2m-1)})=0$.
Thus $C_{1/(2m-1)}, (2\le m<\infty)$ has Lebesgue measure zero.
Hence $C_{1/(2m-1)}, (2\le m<\infty)$ is measurable and has Lebesgue measure zero.

**Proposition 4.6.** Let $(\Gamma_n)$ be an infinite decreasing sequence of each measurable sets $C_{1/(2m-1)}$, that is, a sequence with $\Gamma_{n+1}\subset\Gamma_n$ for each $n$, and $\lambda(\Gamma_1)$ be finite. Then $\lambda\left(\bigcap_{i=1}^{\infty}\Gamma_i\right)=\lim_{n\to\infty}\lambda(\Gamma_n)$ for $C_{1/(2m-1)}$, where $2\le m<\infty$.

**Proof:** Since $(\Gamma_n)_{n\in\mathbf{N}}$ is an infinite decreasing sequence of each measurable set $C_{1/(2m-1)}$, $C_{1/(2m-1)}=\bigcap_{i=1}^{\infty}\Gamma_i$, where $2\le m<\infty$. Let $\Omega_i=\Gamma_i\sim\Gamma_{i+1}$.

Then $\Gamma_1\sim C_{1/(2m-1)}=\bigcup_{i=1}^{\infty}\Omega_i$ and the sets $\Omega_i$ are pair wise disjoint.

$$\therefore\ \lambda(\Gamma_1\sim C_{1/(2m-1)})=\lambda\left(\bigcup_{i=1}^{\infty}\Omega_i\right)=\sum_{i=1}^{\infty}\lambda(\Omega_i)=\sum_{i=1}^{\infty}\lambda(\Gamma_i\sim\Gamma_{i+1})\qquad(1)$$

But we know $\lambda(\Gamma_1)=\lambda(C_{1/(2m-1)})+\lambda(\Gamma_1\sim C_{1/(2m-1)})$, since $C_{1/(2m-1)}\subset\Gamma_1$
and $\lambda(\Gamma_i)=\lambda(\Gamma_{i+1})+\lambda(\Gamma_i\sim\Gamma_{i+1})$, since $\Gamma_{i+1}\subset\Gamma_i$. Since $\lambda(\Gamma_i)\le\lambda(\Gamma_1)<\infty$,
we have $\lambda(\Gamma_1\sim C_{1/(2m-1)})=\lambda(\Gamma_1)-\lambda(C_{1/(2m-1)})$, $\lambda(\Gamma_i\sim\Gamma_{i+1})=\lambda(\Gamma_i)-\lambda(\Gamma_{i+1})$

From (1), we have $\lambda(\Gamma_1)-\lambda(C_{1/(2m-1)})=\sum_{i=1}^{\infty}(\lambda(\Gamma_i)-\lambda(\Gamma_{i+1}))$

$$\lambda(\Gamma_1)-\lambda(C_{1/(2m-1)})=\lim_{n\to\infty}\sum_{i=1}^{n-1}(\lambda(\Gamma_i)-\lambda(\Gamma_{i+1}))=\lambda(\Gamma_1)-\lim_{n\to\infty}\lambda(\Gamma_n)$$

Since $\lambda(\Gamma_1)<\infty$, we have $\lambda(C_{1/(2m-1)})=\lim_{n\to\infty}\lambda(\Gamma_n)$.
Hence $\lambda(C_{1/(2m-1)})=\lim_{n\to\infty}\lambda(\Gamma_n)$ for $C_{1/(2m-1)}$, where $2\le m<\infty$.

**Alternative method:**

**Proposition 4.7.** If $X=[0,1]$ is a closed and $\mathcal{B}$ is a collection of subsets of $X$, then $(X,\mathcal{B})$ is a measurable space. If $\Gamma_i\in\mathcal{B}$, $\mu(\Gamma_1)<\infty$ and $\Gamma_{i+1}\subset\Gamma_i$, then

$\mu\left(\bigcap_{i=1}^{\infty}\Gamma_i\right)=\lim_{n\to\infty}\mu(\Gamma_n)$ for $C_{1/(2m-1)}$, where $2\le m<\infty$.

**Proof:** Since $C_{1/(2m-1)}=\bigcap_{i=1}^{\infty}\Gamma_i$, then $\Gamma_1=C_{1/(2m-1)}\cup\bigcup_{i=1}^{\infty}(\Gamma_i\sim\Gamma_{i+1})$, and this is a disjoint union. Hence $\mu(\Gamma_1)=\mu(C_{1/(2m-1)})+\sum_{i=1}^{\infty}\mu(\Gamma_i\sim\Gamma_{i+1})$ (2)

Since $\Gamma_i=\Gamma_{i+1}\cup(\Gamma_i\sim\Gamma_{i+1})$ is a disjoint union,

we have $\mu(\Gamma_i\sim\Gamma_{i+1})=\mu(\Gamma_i)-\mu(\Gamma_{i+1})$.

Now from (2) we have

$$\mu(\Gamma_1)=\mu(C_{1/(2m-1)})+\sum_{i=1}^{\infty}(\mu(\Gamma_i)-\mu(\Gamma_{i+1}))=\mu(C_{1/(2m-1)})+\lim_{n\to\infty}\sum_{i=1}^{n-1}(\mu(\Gamma_i)-\mu(\Gamma_{i+1}))$$ He

$$=\mu(C_{1/(2m-1)})+\mu(\Gamma_1)-\lim_{n\to\infty}\mu(\Gamma_n)$$

nce $\mu(C_{1/(2m-1)})=\lim_{n\to\infty}\mu(\Gamma_n)$ for $C_{1/(2m-1)}$, where $2\le m<\infty$.

## 5. Concluding remarks

We have shown that generalized Cantor set is measurable set, Borel set as well as Borel measurable whose Lebesgue measure is zero. Also we have proved several interesting lemma, theorems and propositions relating to generalized Cantor set. These results may be extended to Metric space, Banach space and Hilbert space.

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