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The Total Dominating Graph

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Abstract. Let $G = (V, E)$ be a graph. Let S be the set of all minimal total dominating sets of G. The total dominating graph $D_t(G)$ of G is the graph with the vertex set $V \cup S$ in which two vertices u and v are adjacent if $u \in V$ and v is a minimal total dominating set of G containing u. In this paper, some properties of this new graph are obtained. Also characterizations are given for graphs (i) whose total dominating graphs are complete bipartite, (ii) whose total dominating graphs are Eulerian.

Keywords: dominating graph, minimal total dominating set, total dominating graph, Eulerian, Hamiltonian

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1. Introduction

All graphs considered here are finite, undirected without loops and multiple edges. Any undefined term in this paper may be found in [1, 2].

Let $G = (V, E)$ be a graph. A set D of vertices in a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. Recently several domination parameters are given in the books by Kulli in [2, 3, 4].

A set D of vertices in G is a total dominating set of G if every vertex of G is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G. A total dominating set D of G is minimal if for any vertex $v \in D$, $D - \{v\}$ is not a total dominating set of G.

We note that any graph G without isolated vertices has a total dominating set. Thus we consider only graphs without isolated vertices.

The total minimal dominating graph $M_t(G)$ of a graph G is the intersection graph defined on the family of all minimal total dominating sets of vertices of G. This concept was introduced by Kulli and Iyer in [5].

The common minimal total dominating graph $CD_t(G)$ of a graph G is the graph with same vertex set as G with two vertices in $CD_t(G)$ adjacent if there exists a minimal total dominating set in G containing them. This concept was introduced by Kulli in [6].

The dominating graph $D(G)$ of a graph G is the graph with the vertex set $V \cup S$ where S is the set of all minimal dominating sets of G in which two vertices u and v are adjacent if $u \in V$ and v is a minimal dominating set in G containing u. This concept was introduced by Kulli et al in [7]. Many other graph valued functions in domination theory were studied, for example, in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and also graph valued functions in graph theory were studied, for example, in [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

The following will be useful in the proof of our result.

Theorem A. [1, p.66] A nontrivial graph is bipartite if and only if all its cycles are even.

In section 2, we obtain some properties of total dominating graphs.

Traversability of some graph valued functions was studied, for example, in [37, 38, 39, 40]. In section 3, we study traversability of total dominating graphs.

The subdivision of an edge *uv* is obtained by removing an edge *uv*, adding a new vertex *w* and adding edges *uw* and *wv*. The subdivision graph $S(G)$ of a graph G is the graph obtained from G by subdividing each edge of G.

2. Total dominating graphs

The definition of dominating graph of a graph inspired us to define the following graph valued function in domination theory.

Definition 1. Let $G = (V, E)$ be a graph. Let S be the set of all minimal total dominating sets of G. The total dominating graph $D_t(G)$ of G is the graph with the vertex set $V \cup S$ in which two vertices u and v are adjacent if $u \in V$ and v is a minimal total dominating set of G containing u.

Example 2. In Figure 1, a graph G and its total dominating graph $D_t(G)$ are shown.

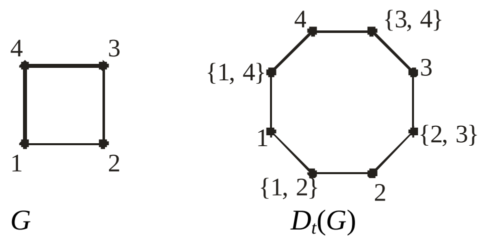

Figure 1

Proposition 3. If G has a vertex which does not lie in any minimal total dominating set, then $D_t(G)$ is disconnected.

Proof: Let u be a vertex of a graph G. If u does not lie in any minimal total dominating set, then u is an isolated vertex in $D_t(G)$. Hence $D_t(G)$ is disconnected.

Theorem 4. If G is a graph without isolated vertices, then $D_t(G)$ is bipartite.

Proof: By definition, no two vertices corresponding to vertices of G in $D_t(G)$ are adjacent and also no two vertices corresponding to minimal total dominating sets of G in $D_t(G)$ are adjacent. Thus $D_t(G)$ has no odd cycles. By Theorem A, $D_t(G)$ is bipartite.

We characterize graphs whose total dominating graphs are complete bipartite.

Theorem 5. The total dominating graph $D_t(G)$ of G is complete bipartite if and only if $G = mK_2$, $m \geq 1$.

Proof: Suppose $D_t(G)$ is complete bipartite. Clearly $V(D_t(G)) = V_1 \cup V_2$, where V_1 is the set of all vertices of G and V_2 is the set of all minimal total dominating sets of G. We now prove that $G = mK_2$, $m \geq 1$. On the contrary, assume $G \neq mK_2$. Then there exists a component G_1 in G which is not K_2. Let v be a vertex of G_1. Then $v \in G$. We consider the following two cases:

Case 1. Suppose $v \notin D$, where D is any minimal total dominating set in G. Then the corresponding vertex of v is an isolated vertex in $D_t(G)$. It implies that the corresponding vertices of D and v are not adjacent in $D_t(G)$. Thus $D_t(G)$ is not complete bipartite, which is a contradiction.

Case 2. Suppose there exist two minimal total dominating sets D_1 and D_2 such that $v \in D_1$ and $v \notin D_2$. Thus the corresponding vertices of v and D_2 are not adjacent in $D_t(G)$. Hence $D_t(G)$ is not complete bipartite, which is a contradiction.

From Case 1 and Case 2, we conclude that every component of G is K_2. Thus $G = mK_2$, $m \geq 1$.

Conversely, suppose $G = mK_2$, $m \geq 1$. Then there exists exactly one minimal total dominating set containing all vertices of G. Then $|V(D_t(G) | = 2m +1$. Thus by definition $D_t(G) = K_{1,\, 2m}$ and hence $D_t(G)$ is complete bipartite.

We also prove the following result.

Theorem 6. $D_t(G)= K_{1,2m}$ if and only if $G = mK_2$, $m \geq 1$.

Proof: Suppose $D_t(G) = K_{1,\, 2m}$, $m \geq 1$. Then $D_t(G)$ is complete bipartite. By Theorem 5, $G = mK_2$, m ≥ 1.

Conversely, suppose $G = mK_2$, m ≥ 1. Then there exists exactly one minimal total dominating set containing all vertices of G. Thus by definition, $D_t(G) = K_{1,\, 2m}$.

The double star $S_{m,\, n}$ is the graph obtained from joining the centers of two stars $K_{1,\, m}$ and $K_{1,n}$ with an edge. The centers of $K_{1,m}$ and $K_{1,\, n}$ are called central vertices of $S_{m,\, n}$. Thus $S_{m,\, n}$ has $m+n+2$ vertices.

Theorem 7. If $S_{m,\, n}$, $1 \leq m \leq n$, is a double star, then

$$D_t(S_{m,n}) = (m+n)\, K_1 \cup K_{1,2}.$$

Proof: Let $S_{m,\, n}$ be a double star, $1 \leq m \leq n$ and u and v be central vertices of $S_{m,\, n}$. Then $S_{m,\, n}$ has exactly one minimal total dominating set D containing the central vertices u and v of $S_{m,\, n}$. Then $D = \{u, v\}$. Thus the vertex set of $D_t(S_{m,\, n})$ in $V \cup D$, where V is a vertex set of $S_{m,\, n}$, and hence $D_t(S_{m,n})$ has $m+n+2+1$ vertices. The corresponding vertices of D and u are adjacent and also the corresponding vertices of D and v are adjacent in $D_t(S_{m,\, n})$ and all other vertices of $D_t(S_{m,\, n})$ are isolated vertices. Thus $D_t(S_{m,\, n})$ is disconnected and

$$D_t(S_{m,n}) = (m+n)K_1 \cup K_2.$$

Theorem 8. Let G be a nontrivial connected graph. Let $S(G)$ be the subdivision graph of G. The graphs $D_t(G)$ and $S(G)$ are isomorphic if and only if every pair of vertices forms a minimal total dominating set of G.

Proof: Let G be a nontrivial connected graph. Suppose $D_t(G) = S(G)$. Since G is connected, $S(G)$ is connected. For each edge $e_i=u_iv_i$ of G, w_i is a new vertex such that u_iw_i and w_iv_i are edges of $S(G)$. Since $D_t(G)= S(G)$, it implies that every pair of vertices u_i,v_i forms a minimal total dominating set of G.

Conversely, suppose every pair of vertices of G forms a minimal total dominating set of G. Then they are adjacent in G. Clearly for each minimal total dominating set D of G, the corresponding vertex of D in $D_t(G)$ is adjacent with exactly two vertices and hence we see that $D_t(G) = S(G)$.

Corollary 9. If $G=K_p$, $p\geq 2$ or $K_{m,n}$, $1\leq m\leq n$, then $D_t(G) = S(G)$.

3. Traversabiltiy

We need the following result.

Theorem B. A connected graph G is Eulerian if and only if every vertex of G has even degree.

We characterize total dominating graphs which are Eularian.

Theorem 10. Let G be a nontrivial connected graph. The total dominating graph $D_t(G)$ of G is Eularian if and only if the following conditions hold:

i) every minimal total dominating set contains even number of vertices,

ii) every vertex of G is in even number of minimal total dominating sets of G.

Proof: Suppose $D_t(G)$ is Eulerian. On the contrary, if condition (i) is not satisfied, then there exists a minimal total dominating set containing odd number of vertices and hence $D_t(G)$ has a vertex of odd degree. By Theorem B, $D_t(G)$ is not Eulerian, a contradiction. Similarly we can prove (ii).

Conversely, suppose the given conditions are satisfied. Then the degree of each vertex in $D_t(G)$ is even. Hence by Theorem B, $D_t(G)$ is Eulerian.

Theorem 11. Let $\Gamma_t(G)$ =2. If every vertex is in exactly two minimal total dominating sets of G, then $D_t(G)$ is Hamiltonian.

Proof: Clearly $\gamma_t(G) = \Gamma_t(G)$ and $D_t(G)$ is connected. Let $v \in V$ and D be a minimal total dominating set of G. Then $\deg_{D(G)} v = \deg_{(D(G)} D = 2$. Hence $D_t(G)$ is connected 2 - regular.

Thus $D_t(G)$ is Hamiltonian.

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