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Results on β_M-Excellent Graphs

J. Joseline Manora[1] **and *B. John***[2]

[1]PG & Research Department of Mathematics,
T.B.M.L College, Porayar,Nagapattinam Dt., Tamilnadu, India
[2]Department of Science and Humanities
E.G.S.Pillay Engineering College
Nagapattinam Dt., Tamilnadu, India
Corresponding author. Email: johnvisle@gmail.com

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Abstract. A graph G is β_M-excellent if every vertex of G is contained in a maximal majority independent set of a graph G .In this paper, we study some standard graphs G which are β_M-excellent and not β_M-excellent graphs. Also we have obtained some results on β_M-excellent graphs in the case of a Cartesian product, generalization of Petersen graph and trees.

Keywords: Majority independence number- $\beta_M(G)$, β_M excellent graphs

AMS Mathematics Subject Classification (2010): 05C69

1. Introduction

Claude Berge in 1980, introduced B graphs. These are graphs in which every vertex in the graph is contained in a maximum independent set of the graph. Fircke et al. [1] in 2002 made a beginning of the study of graphs which are excellent with respect to various parameters. γ-excellent trees and total domination excellent trees have been studied in [1].Also Sridharan and Yamuna [9]made an extensive work in this area. Swaminathan and Pushpalatha have defined β_o-excellent graphs, just β_o-excellent graphs and very β_o-excellent graphs and they have made a detailed study in this paper [8].

By a graph G , we mean a finite, simple graph which is undirected and nontrivial. Let $G=(V,E)$ be a graph of order p and size q. For every vertex $v\in V(G)$, the open neighbourhood $N(v)=\{u\in V(G)/uv\in E(G)\}$ and the closed neighbourhood $N[v]=N(v)\bigcup\{v\}$. Let S be a set of vertices, and let $u\in S$. The private neighbour set of u with respect to S is $pn[u,S]=\{v/N[v]\bigcap S=\{u\}\}$

Definition 1.1. [2] A set D of vertices in a graph G is called an independent set if no two vertices in D are adjacent. An independent set D is called a maximal independent set if any vertex set properly containing D is not independent. The independence number $\beta_o(G)$ is the maximum cardinality of a maximal independent set in G. Let $G=(V,E)$ be a simple graph. Let $u\in V(G)$. The vertex u is said to be β_o-good if u is contained in a β_o-set of G. The vertex u is said to be β_o-bad if there exists no β_o-set of G containing u [8].

Definition 1.2. [4] A set S of vertices of a graph G is said to be a Majority Independent set(or MI-set) if it induces a totally disconnected subgraph with $|N[S]|\geq\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,S]|>|N[S]|-\left\lceil \frac{p}{2}\right\rceil$ for every $v\in S$. If any vertex set S' properly containing S is not majority independent. Then S is called Maximal Majority Independent set. The minimum cardinality of a maximal majority independent set is called lower majority independence number of Gand it is also called Independent Majority Domination number of G. It is denoted by $i_M(G)$. The maximum cardinality of a maximal majority independent set of G is called Majority Independence number of G and it is denoted by $\beta_M(G)$. A β_M-set is a maximum cardinality of a maximal majority independent set of G. This parameter has been studied by Swaminathan and Joseline Manora.

2. Majority independence number of some graphs

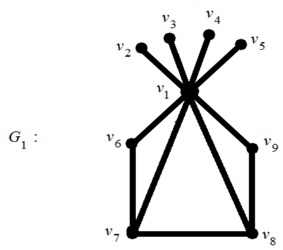
We have determined $\beta_M(G)$ for several standard graphs in [4]. Here we consider some classes of graphs which are needed for our study and compute its $\beta_M(G)$.

1. Let $G=P_p$, $p\geq 2$. Then $\beta_M(G)=\begin{cases}\left\lceil \frac{p}{4}\right\rceil & if\ \ p\leq 6.\\ \left\lceil \frac{p-2}{4}\right\rceil & if\ \ p>6.\end{cases}$
2. Let $G=K_{1,p-1}$, $p\geq 2$. Then $\beta_M(G)=\left\lfloor \frac{p-1}{2}\right\rfloor$.
3. Let $G=W_p$, $p\geq 5$. Then $\beta_M(G)=\left\lceil \frac{p-2}{6}\right\rceil$.
4. Let $G=D_{r,s}$, $r,s\geq 2$. Then $\beta_M(G)=\begin{cases} r & if\ \ r=s,\ p=r+s+2.\\ \left\lceil \frac{p}{2}\right\rceil-1 & if\ \ r<s.\end{cases}$
5. Let $G=\overline{K_p}$, $p\geq 2$ Then $\beta_M(G)=\left\lceil \frac{p}{2}\right\rceil$.

3. β_M-Excellency on some standard graphs

Definition 3.1. [3] Let $G=(V,E)$ be a simple graph. Let $u \in V(G)$. The vertex u is said to be β_M-good if u is contained in a β_M-set of G. The vertex u is said to be β_M-bad if there exists no β_M-set of G containing u. A graph G is said to be β_M-excellent if every vertex of G is β_M-good.

Example 3.2. In the following graphs G_1 and G_2,



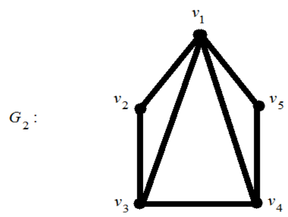


Figure 1:

The vertices v_1, v_6, v_7, v_8 and v_9 are β_M-bad vertices in G_1. $\therefore G_1$ is not β_M-excellent.

All vertices are β_M-good vertices in G_2. Therefore, G_2 is β_M-excellent.

Results 3.3. Some standard β_M-excellent graphs

1. K_p, $p \geq 2$ is β_M-excellent.
2. C_p, $p \geq 3$ is β_M-excellent.
3. $K_{m,n}$ is β_M-excellent if $m=n$.
4. $\overline{K_p}$ is a β_M-excellent graph.
5. The tri-partite graph $K_{m,n,r}$ is β_M-excellentif $m=n=r$ otherwise not β_M-excellent.
6. Let $G=C_n \square C_m$, $m \geq 3$. Then the Torus is β_M-excellent.

Results 3.4. Some examples for not β_M-excellent graphs

1. $K_{1,p-1}$, $p \geq 4$ is not β_M-excellent.
2. Let $G=P_n \square P_m$, $n,m \geq 2$. Then the Grid graph is not β_M-excellent.
3. The Wounded spider and a binary tree are not β_M-excellent.
4. The Caterpillar is not β_M-excellent.
5. $K_{m,n}$, $m<n$ is not β_M-excellent.

6. If $G = D_{r,s}$, $r, s \geq 1$, then G is not β_M -excellent.

Theorem 3.5. [4] Let G be a cycle of p vertices, $p \geq 3$ .Then $\beta_M(G) = \left\lceil \frac{p}{6} \right\rceil$.

Theorem 3.6. If $G = C_p$, $p \geq 3$, then G is β_M -excellent.

Proof: When $p = 3,4,5,6$. Since $\beta_M(G) = \left\lceil \frac{p}{6} \right\rceil$, every single vertex is a β_M -set and all vertices are β_M -good. Then G is β_M -excellent.

Case (i): When $p = (n+6)$, $n = 1,2,...,6 \Rightarrow p = 7,8,9,10,11,12$. Since $\beta_M(G) = \left\lceil \frac{p}{6} \right\rceil$, β_M -sets are $\{v_i, v_{i+3 \pmod p} \ ; i = 1,..., p\}$. All vertices of G belong to any one of the β_M -sets of G $\therefore G$ is β_M -excellent.

Case (ii): When $p = (n+6)$, $n = 7,8,...,12 \Rightarrow p = 13,14,15,16,17,18$. Since $\beta_M(G) = \left\lceil \frac{p}{6} \right\rceil = 3$, β_M -sets are $\{v_i, v_{i+3 \pmod p}, v_{i+6 \pmod p} \ ; \ i = 1, 2, ..., p\}$.

Case (iii): When $p = (n+6)$, $n = 13, 14, ..., 18 \Rightarrow p = 19, 20, 21, 22, 23, 24$. Since $\beta_M(G) = \left\lceil \frac{p}{6} \right\rceil = 4$, β_M -sets are

$$\{v_i, v_{i+3 \pmod p}, v_{i+6 \pmod p}, v_{i+9 \pmod p} \ ; \ i = 1, 2, ..., p\}.$$

In general, when $p = (n-5), (n-4), ..., n$. Since β_M -sets of G consist of $\left\lceil \frac{p}{6} \right\rceil$ vertices, β_M -sets are $\left\{v_i, v_{i+3 \pmod p}, v_{i+6 \pmod p}, ..., v_{i+3r \pmod p} \ ; \ i = 1, 2, ..., p \ \ and \ \ r = 0,1,2,..., \left\lceil \frac{p}{6} \right\rceil - 1\right\}$.

$\therefore$ All vertices in G are β_M -good. Hence $G = C_p$, $p \geq 3$ is β_M -excellent.

4. Results on β_M -excellent graphs

Proposition 4.1. If $G \neq K_p$ has a full degree vertex and all other vertices are of degree $< \left\lceil \frac{p}{2} \right\rceil - 1$, then G is not β_M -excellent.

Definition 4.2. [6] If the degree of a vertex $v \in V(G)$ satisfies the condition $d(v) \geq \left\lceil \frac{p}{2} \right\rceil - 1$, then the vertex v is called a majority dominating vertex of G.

Observation 4.3. If G has all vertices of majority dominating vertices then G is β_M-excellent.

Proposition 4.4. Suppose G has a majority dominating vertex, then G is not β_M-excellent.

Proof: Suppose G has a majority dominating vertex u and other vertices of degree $d(v_i) < \left\lceil \frac{p}{2} \right\rceil - 1$, $i = 1, 2, ..., p-1$. Then u is a β_M-bad vertex and v_i, $i = 1, 2, ..., p-1$ are β_M-good vertices. $\therefore$ G is not β_M-excellent.

Theorem 4.5. Suppose G has $pC_{\left\lceil \frac{p}{2} \right\rceil}$ - β_M sets. Then G is β_M-excellent if and only if $G = \overline{K_p}$.

Proof: Assume that G is β_M-excellent $\Rightarrow$ All vertices of G are β_M-good $\Rightarrow$ All vertices of $V(G)$ must belong to any one of the majority independent set of G. Since G has $pC_{\left\lceil \frac{p}{2} \right\rceil}$ - β_M sets, each majority independent set D contains $\left\lceil \frac{p}{2} \right\rceil$ vertices and also each β_M-set is a combination of $\left\lceil \frac{p}{2} \right\rceil$ vertices of $V(G)$. Then, each β_M-set consists of only isolates of G and $|D| = \left\lceil \frac{p}{2} \right\rceil$. $\therefore$ The graph is disconnected with isolates.

Suppose $G = K_2 \cup \overline{K_{p-2}}$, then G contains exactly one edge and others are isolates. In G, every maximal majority independent set D contains only isolates and the combination of $\left\lceil \frac{p}{2} \right\rceil$ vertices of $\overline{K_{p-2}}$ is also a β_M-set of G. Thus, we have obtained that there are $(p-2)C_{\left\lceil \frac{p}{2} \right\rceil}$ β_M-sets for the graph G and vertices of K_2 are β_M-bad vertices. It gives a contradiction. Hence, if G is β_M-excellent with $pC_{\left\lceil \frac{p}{2} \right\rceil}$ - β_M sets then $G = \overline{K_p}$. The converse is obvious.

Theorem 4.6. Let $G = P_p$, $p \geq 2$. Then

(i) G is not β_M-excellent if $p = 5, 6$.

(ii) G is β_M-excellent if $p < 13$.

(iii) Let $p \geq 13$. If $p \equiv 1,2,5,6 \pmod 8$, G is β_M -excellent.

(iv) If $p \equiv 0,3,4,7 \pmod 8$ and $p \geq 15$, G is not β_M -excellent.

Proposition 4.7. If a tree T with more than two pendants with $p \geq 4$, then T is not β_M -excellent.

Proof: Let e be the number of pendants in T. If T has $p = 4$ and $e = 3$ pendants then $T = K_{1,3}$. Suppose T has $p = 5$ and $e = 4$ pendants, then $T = K_{1,4}$ or T has an induced subgraph $K_{1,3}$. With the same argument, Suppose T has $(p-1)$ vertices and $3 \leq e \leq (p-2)$ pendants, then $T = K_{1,p-2}$ or T has an induced subgraph of $K_{1,e}$, where $3 \leq e \leq (p-3)$. If T has p vertices and $3 \leq e \leq (p-1)$ pendants then $T = K_{1,p-1}$ or T has an induced subgraph of $K_{1,e}$, where $3 \leq e \leq (p-2)$.By result (3.4.1), we found that every $K_{1,p-1}$, $p \geq 4$ is not β_M -excellent. Hence, the tree T with more than two pendants is not a β_M -excellent graph.

Corollary 4.8. If a tree T has a vertex v with $d(v) \geq 3$ then T is not β_M -excellent.

Definition 4.9. [8] For each $n \geq 3$ and $0 < k < n$, $P(n,k)$ denotes the generalized Petersen graph with vertex set $V(G) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and the edge set $E(G) = \{u_i u_{i+1 \pmod n}, u_i v_i, v_i v_{i+k \pmod n}\}$, $1 \leq i \leq n$.

Theorem 4.10. Let G be a generalization of Petersen graph $P(n,k)$ with $k=1$, $n \geq 3$.

Then $\beta_M(G) = \begin{cases} \left\lceil \dfrac{p-4}{4} \right\rceil & if\ n<7 \\ \left(\dfrac{p}{7}\right) & if\ n=7 \\ \left\lfloor \dfrac{p-3}{4} \right\rfloor & if\ n \geq 8 \end{cases}$ and also $G = P(n,1)$ is β_M -excellent.

Proof: Let G be a generalized Petersen graph $P(n,1)$ with $|V(G)| = 2n = p$ vertices. Then G consists of two cycles C_1 and C_2 such that the cycle C_1 with vertex set $\{v_1, v_2, ..., v_n\}$ is nested by the another cycle C_2 with vertex set $\{u_1, u_2, ..., u_n\}$ and each u_i in C_2 is incident with exactly one v_i in C_1 and $d(v_i) = d(u_i) = 3$, $i = 1, 2, ..., n$.

Case (i): When $n < 7$. The maximum majority independent sets are $\{v_i, u_{i+1 \,(\mathrm{mod}\, n)}\}$, $i = 1, 2, ..., n$. Then $\beta_M(G) = 2 = \left\lceil \frac{p-4}{4} \right\rceil$, if $n < 7$.

Case (ii): When $n = 7$. The maximum majority independent sets are $\{v_i,\ u_{i+2 \,(\mathrm{mod}\, n)},\ i = 1, 2, ..., 7\}$. $\therefore \beta_M(G) = 2 = \left(\frac{p}{7}\right)$.

Case (iii): When $n \geq 8$. Let $D = \{u_1, u_2, .., u_t\}$, $t = \left\lfloor \frac{p-3}{4} \right\rfloor$ and $d(u_i, u_j) \geq 2$, $i \neq j$. Then $|N[D]| = \sum_{i=1}^{t}(d(u_i)+1) = 4t = 4\left\lfloor \frac{p-3}{4} \right\rfloor \geq \left\lceil \frac{p}{2} \right\rceil$. Also, for every $v \in D$, $|pn[v,D]| > |N[D]| - \left\lceil \frac{p}{2} \right\rceil$. Hence D is a β_M-set of G. $\therefore \beta_M(G) \geq |D| = \left\lfloor \frac{p-3}{4} \right\rfloor$. Suppose $S = \{v_1, v_2, .., v_r\}$, $r = \left\lfloor \frac{p-3}{4} \right\rfloor + 1$ with $d(v_i, v_j) \geq 2$, $i \neq j$. But $|pn[v,S]| \leq |N[S]| - \left\lceil \frac{p}{2} \right\rceil$, for any $v \in S$. $\therefore S$ is not a β_M-set of G. Hence $\beta_M(G) < |S| = \left\lfloor \frac{p-3}{4} \right\rfloor + 1 \Rightarrow \beta_M(G) \leq \left\lfloor \frac{p-3}{4} \right\rfloor$. Therefore $\beta_M(G) = \left\lfloor \frac{p-3}{4} \right\rfloor$. The maximum majority independent sets are $\{v_i,\ u_{i+1 \,(\mathrm{mod}\, n)},\ v_{i+2 \,(\mathrm{mod}\, n)}, u_{i+3 \,(\mathrm{mod}\, n)}, ...\}$, $\{u_i,\ v_{i+1 \,(\mathrm{mod}\, n)}, u_{i+2 \,(\mathrm{mod}\, n)}, v_{i+3 \,(\mathrm{mod}\, n)}, ...\}$, $i = 1, 2, ..., n$. In all the above cases, all vertices of $V(G)$ are β_M-good. Hence $G = P(n, 1)$ is β_M-excellent.

Remark 4.11. We have determined that the most popular regular graphs such as complete graph, cycle, generalization of Petersen graph and $K_{m,n}$ if $m = n$ are β_M excellent graphs. But all regular graphs are not β_M-excellent.

Example 4.12. This graph G_3 is a cubic graph. Here $\{v_1, v_2, v_5, v_7, v_9, v_{11}, v_{13}, v_{16}\}$ are β_M-good vertices and $\{v_3, v_4, v_6, v_8, v_{10}, v_{12}, v_{14}, v_{15}\}$ are β_M-bad vertices. Therefore G_3 is not β_M-excellent.

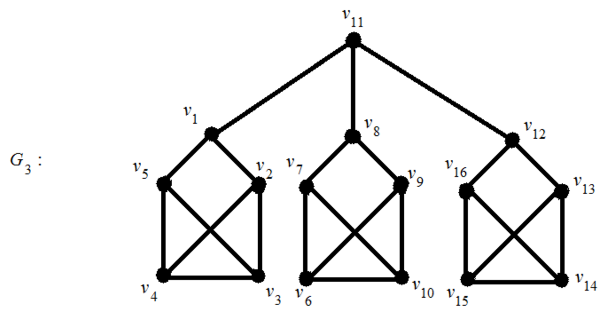


Figure 2:

Theorem 4.13. Let H be a graph without pendants. Then $\overline{K_p} \square H$ is β_M -excellent if and only if H is β_M -excellent.

Proof: Let H be a β_M -excellent graph. Let $\{u_1, u_2, ..., u_q\}$ be the vertices of H. The construction of $\overline{K_p} \square H$ is the graph of H in p times. Since all vertices of H are β_M -good, all vertices of the graph $\overline{K_p} \square H$ are β_M -good. Therefore, $\overline{K_p} \square H$ is β_M -excellent. Suppose H is not β_M -excellent then any vertex $u \in V(H)$ is not contained in any β_M -set of H. Suppose S is a β_M -set of $\overline{K_p} \square H$ containing (v, u) for some $v \in V(\overline{K_p})$, then S is of the form $V(\overline{K_p}) \times A$, where A is a β_M -set of H. Therefore $u \in A$, which is a contradiction.

Theorem 4.14. Let H be a graph without pendants. Then $K_n \square H$ is β_M -excellent if and only if H is β_M -excellent.

Proposition 4.15. Let $G = P_2 \square P_m$. Then $\beta_M(G) = \begin{cases} \left\lfloor \frac{p}{4} \right\rfloor & if \quad p \le 10 \\ \left\lceil \frac{p-4}{4} \right\rceil & if \quad p > 10 \end{cases}$

Theorem 4.16. If $G = P_2 \square P_m$, then the following results are true in G.

(i) When $m=2,3,5$, G is β_M -excellent.

(ii) When $m=4$, G is not β_M -excellent.

(iii) When $m>6$ and m is odd, G is not β_M -excellent.

(iv) If m is even and $m \ge 6$, G is β_M -excellent.

Proof: Let $G = P_2 \square P_m$. Here $\Delta(G)=3$ and $\delta(G)=2$. Then $V(P_2)=(u_1, u_2)$ and $V(P_m)=\{v_1, v_2, ... v_m\}$.

Case (i): When $m=2,3$. Then every vertex in G is a β_M -set of G . $\therefore$ G is β_M -excellent.

When $m=5$. The β_M -sets are $(u_1v_1, u_2v_{i+1}), (u_2v_1, u_1v_{i+1})$, $i=1,2,3,4$. Also, (u_1v_1, u_1v_{i+2}) and (u_2v_1, u_2v_{i+2}), $i=1,2,3$. Hence G is β_M -excellent.

Case (ii): When $m=4$ .The β_M -sets are $(u_1v_1, u_2v_{i+2}), (u_2v_1, u_1v_{i+2})$, $i=1,2$ and also (u_1v_1, u_1v_4), (u_2v_1, u_2v_4). In $V(G)$, $(u_1v_i), (u_2v_i)$, $i=2,3$ are β_M -bad vertices of G . $\therefore$ G is not β_M -excellent.

Case (iii): When $m>6$ and m is odd. Here, there are some β_M -bad vertices such as $(u_1v_i), (u_2v_i)$, $i=4,5,...$ The remaining vertices are β_M -good vertices. Hence G is not β_M -excellent.

Case (iv): $m\geq 6$ and m is even. When $m=6$, $i=1,2,3,4$. Then the β_M -sets are (u_1v_1, u_2v_{i+2}), (u_1v_1, u_1v_{i+2}), (u_2v_1, u_1v_{i+2}), (u_2v_1, u_2v_{i+2}).

When $m=8$, $i=1,2,3,4$. The β_M -sets are $(u_1v_1, u_2v_{i+2}, u_2v_{i+4})$, $(u_1v_1, u_1v_{i+2}, u_1v_{i+4})$, $(u_2v_1, u_1v_{i+2}, u_1v_{i+4})$, $(u_2v_1, u_2v_{i+2}, u_2v_{i+4})$.

In general, $m>8$, $i=1,2,3,4$. The β_M -sets are $(u_1v_1, u_2v_{i+2}, u_2v_{i+4}, \ldots)$, $(u_1v_1, u_1v_{i+2}, u_1v_{i+4}, \ldots)$, $(u_2v_1, u_1v_{i+2}, u_1v_{i+4}, \ldots)$, $(u_2v_1, u_2v_{i+2}, u_2v_{i+4}, \ldots)$. Since every vertex of G belongs to any β_M -set of G . Hence G is β_M -excellent.

REFERENCES

1. G.H.Fricke, T.W.Haynes, S.T.Hadetniemi, S.M.Hedetniemi and R.C.Laskar, Excellent trees, *Bull. Inst. Combin. Appl.*, 34 (2002) 27-38.
2. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, Fundamentals of domination in graphs, Marcel Dekkar, New York, 1998.
3. J.JoselineManora and B.John, $\beta_M(G)$ -excellent graphs, *Proceedings of the International Conference on Jamal Research Journal: An Interdisciplinary (Special Issue)*, (2015) 513-517.
4. J.Joseline Manora and B.John, Majority independence number of a graph, *International Journal of Mathematical Research*, 6(1) (2014) 65-74.
5. J.Joseline Manora andI.Paulrajjayasimman, Neighborhood sets and neighborhood polynomials of cycles, *Annals of Pure and Applied Mathematics*, 7(2) (2014) 45-51.

6. J.Joseline Manora and V.Swaminathan, Results on majority dominating set, *Science Magna*, 7(3) (2011) 53-58.
7. J.Joseline Manora and S.Veeramanikandan, The split majority domination number of a graph, *Annals of Pure and Applied Mathematics*, 9(1) (2015) 13-22.
8. A.P.Pushpalatha, G.Jothilakshmi, S.Suganthi and V.Swaminathan, β_o -excellent graphs, *WSEAS Transaction on Mathematics*, 10(2) (2011) .
9. N.Sridharan and M.Yamuna, A note on excellent graphs, *ARS Combinatoria*, 78 (2006) 267-276.