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Norm and Metric Defined Over Fuzzy Sets

P.A. Ejegwa **and** ***J. M. Agbetayo***

Department of Mathematics/Statistics/Computer Science
University of Agriculture, P.M.B. 2373, Makurdi-Nigeria
Corresponding author. Email: ocholohi@gmail.com

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Abstract.We reviewed some preliminaries of fuzzy sets and deduced some relevant theorems with proofs. We then defined the concept of norm and metric on fuzzy sets and proved some results.

Keywords: Fuzzy sets, norm, metric

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1. Introduction

Fuzzy set is a class in which there may be a continuum of grades of membership as, say, in the class of *long* objects. Such sets underlie much of our ability to summarize, communicate, and make decisions under uncertainty or partial information. Indeed, fuzzy sets appear to play an essential role in human cognition, especially in relation to concept formation, pattern classification, and logical reasoning.Since the introduction of fuzzy sets by Zadeh [3] as the extension of crisp sets, the theory of fuzzy sets has evolved in many directions, and is finding applications in a wide variety of fields in which the phenomena under study are too complex or too ill defined to be analyzed by conventional techniques.

Thus, by providing a basis for a systematic approach to approximate reasoning, the theory of fuzzy sets well have a substantial impact on scientific methodology, particularly in the realms of psychology, economics, law, medicine, decision analysis, information retrieval, artificial intelligence, mathematics, control and expert systems, possibility theory, environmental pollution control, human recognition, communication, fuzzy logic etc. [1, 2, 4, 5]. In terms of the analysis, see [7, 8] where coupled coincidence point results in G-complete fuzzy metric space and N-topological spaces associated with fuzzy topological spaces were discussed. Gebray and Reddy [9] introduced metric on a subset of a fuzzy linear space and Goleţ [10] generalized fuzzy 2-norms on a set of objects endowed with a structure of linear space.

In this paper, we present some basics of fuzzy sets theory, deduce some results with proofs. After ward, define the concept of norm and metric on fuzzy sets axiomatically, and deduce some important results.

2. Precise note on fuzzy sets

Definition 1. Crisp set A of X is defined as the characteristic function of A and is denoted by $f_A(x)$, mathematically, $f_A(x): X \to \{0,1\}$

where, $f_A(x) = \begin{Bmatrix} 1, if\ x \in A \\ 0, if\ x \notin A \end{Bmatrix}$

Definition 2. Fuzzy set A of a set X is defined by the membership function of the set $s.t.\ \mu_A(x): X \to [0,1]$,

where, $\mu_A(x) = \begin{Bmatrix} 1,\ if\ x\ is\ totally\ in\ A \\ 0, if\ x\ is\ not\ in\ A \\ (0,1), if\ x\ is\ partly\ in\ A \end{Bmatrix}$

The closer the membership value $\mu_A(x)$ to 1, the more x belongs to A, where the grades 1 and 0 represents full membership and full non-membership.

Fuzzy set is a collection of objects with graded membership i.e. having degrees of membership. Fuzzy set is an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition; an element either belongs or does not belong to the set. Classical bivalent sets are in fuzzy set theory called crisp sets. Fuzzy sets are generalized classical sets, since the indicator function of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1.

Definition 3. Let X be nonempty and A be fuzzy set in X, then;
(i) $Supp(A) = \{x \in X: \mu_A(x) > 0\}$ where $Supp(A)$ is the support of A.
(ii) The crossover point of A is $\{x \in X: \mu_A(x) = 0.5\}$.
(iii) $hgt(A) = sup\ \mu_A(x)$ where $hgt(A)$ is the height of A.
(iv) A is normalized if $hgt(A) = 1$.

Definition 4. Let X be nonempty and A and B be fuzzy sets in X, then;
(i) $\mu_{A\cup B}(x) = max\ (\mu_A(x), \mu_B(x))$ where $\mu_{A\cup B}(x)$ is the union of fuzzy sets A and B.
(ii) $\mu_{A\cap B}(x) = min\ (\mu_A(x), \mu_B(x))$ where $\mu_{A\cap B}(x)$ is the intersection of fuzzy sets A and B.
(iii) $\mu_{A'}(x) = 1 - \mu_A(x)$ is the compliment of A.
(iv) $\mu_A(x) = \mu_B(x)$, then A and B are equal.

Definition 5. Let X be nonempty and A be a fuzzy set whose membership degree $\mu_A(x_i)$ is a function s.t. $\mu_A(x_i): X \to [0,1]$ for $X = \{x_1, \ldots, x_n, \ldots\}$. Then A is aninfinite fuzzy set. On the other hand, if $X = \{x_1, \ldots, x_n\}$ and $\mu_A(x_i)$ takes each x_i for $i = 1, \ldots, n$ to form image in $[0,1]$, then A is called finite.

Definition 6. Let A and B be two fuzzy sets drawn from a nonempty X. A is a fuzzy subset of B denoted by $A \subseteq B$ if $\mu_A(x_i) \subseteq \mu_B(x_i)$, and so B is a fuzzy superset of A for $i = 1, \ldots, n$.

Remarks (i) Every fuzzy set contains itself.

(ii) If $A = B \Longrightarrow A$ is a fuzzy subset of B and B is a fuzzy subset of A.

Definition 7. Fuzzy set A is a proper fuzzy subset of B denoted by $A \subset B$ if $A \subseteq B$ and $A \neq B$ i.e. $\mu_A(x_i) \subseteq \mu_B(x_i)$ and $\mu_A(x_i) \neq \mu_B(x_i)$ $i = 1, \dots, n$..

Definition 8. Two fuzzy sets A and B are comparable if $A \subset B$ or $B \subset A$.

Theorem 1. If A is a fuzzy subset of B and B is a fuzzy subset of C, then A is a fuzzy subset of C.

Proof. Let $\alpha \in \mu_A(x)$ for $\alpha \in [0,1]$. Since $\mu_A(x) \subseteq \mu_B(x)$, then α is also in B i.e. $\alpha \in \mu_B(x)$. But by hypothesis, $\mu_B(x) \subseteq \mu_C(x)$, hence every member of B including α is also in C. Obviously, $\alpha \in \mu_A(x) \Longrightarrow \alpha \in \mu_C(x)$. So $A \subseteq B$ and $B \subseteq C \Longrightarrow A \subseteq C$.□

Theorem 2. Let A and B be two fuzzy sets s.t. $A \subset B$, then (i) $A \cap B = A$ (ii) $A \cup B = B$ (iii) $B' \subset A'$.

Proof. (i) Recall that $A \cap B = \min(\mu_A(x), \mu_B(x))$ and $A \subset B \Longrightarrow \mu_A(x) \subseteq \mu_B(x)$ and $\mu_A(x) \neq \mu_B(x)$. Then $A \cap B = \min\left(\mu_A(x), \mu_B(x)\right) = \mu_A(x) = A$.

(ii) Again, $A \cup B = \max(\mu_A(x), \mu_B(x))$ and $\mu_A(x) \subseteq \mu_B(x)$ and $\mu_A(x) \neq \mu_B(x) \Longrightarrow A \cup B = \mu_B(x) = B$.

(iii) We know that $A' = 1 - \mu_A(x)$ and $B' = 1 - \mu_B(x)$. Since $A \subset B, 1 - \mu_B(x) \subseteq 1 - \mu_A(x)$ and $\mu_A(x) \neq \mu_B(x)$. We conclude that $B' \subset A'$.□

3. Norm defined over fuzzy sets

Definition 9. Let $\mu_A(x_i)$ be a membership function of a fuzzy set A drawn from non-empty set X s.t. $X = \{x_1, \dots, x_n\}$ for $i = 1, \dots, n$. A norm on $\mu_A(x_i)$ is a function $\|.\|: X \to [0,1]$ that satisfies the following conditions:

(i) $\|\mu_A(x_i)\| \geq 0\ \forall\ A \in X$ (nonnegative), and $\|\mu_A(x_i)\| = 0 \Leftrightarrow \mu_A(x_i) = 0$ (strictly positive);

(ii) $\|\alpha\mu_A(x_i)\| = |\alpha|\,\|\mu_A(x_i)\|, \forall A \in X$ and $\alpha \in [0,1]$ (homogeneous);

(iii) $\|\mu_A(x_i) + \mu_B(x_i)\| \leq \|\mu_A(x_i)\| + \|\mu_B(x_i)\|\ \forall\, A, B \in X$.

The fuzzy set equipped with a norm is called a normed fuzzy set. The condition (iii) is called the triangle inequality.

Lemma 1. Let $\mu_A(x_i)$ be a membership function of a normed fuzzy set A. Then $\|(x_i) - \mu_B(x_i)\| \geq \left| \|\mu_A(x_i)\| - \|\mu_B(x_i)\| \right|, \forall\, A, B \in X$ for $i = 1, \dots, n$. This lemma is called reverse triangle inequality.

Proof. Let $A, B \in X$. We know that; $\|\mu_A(x_i) - \mu_B(x_i)\| \geq \left| -(-\|\mu_A(x_i)\| + \|\mu_B(x_i)\|) \right| = \left| \|\mu_B(x_i)\| - \|\mu_A(x_i)\| \right|$ i.e.

$$\|\mu_A(x_i) - \mu_B(x_i)\| \geq \|\mu_B(x_i)\| - \|\mu_A(x_i)\| \text{or} \tag{1}$$

$$\|\mu_A(x_i) - \mu_B(x_i)\| \geq \|\mu_A(x_i)\| - \|\mu_B(x_i)\| \tag{2}$$

The proofs of (1) and (2) are similar, so we prove only (2). Using condition (iii) above, $\|\mu_A(x_i)\| = \|\|\mu_A(x_i) - \mu_B(x_i) + \mu_B(x_i)\| = \|\left(\mu_A(x_i) - \mu_B(x_i)\right) + \mu_B(x_i)\| \leq$

$\| \mu_A(x_i) - \mu_B(x_i)\| + \|\mu_A(x_i)\| \Rightarrow \| \mu_A(x_i)\| - \|\mu_B(x_i)\| \geq \| \mu_A(x_i) - \mu_B(x_i)\|$ as desired. □

4. Metric defined over fuzzy sets

Definition 10. Let X be an arbitrary non-empty set. A metric, or distance function on X is a function $d: X \times X \to [0, 1]$ with the following properties:

(i) $d(\mu_A(x_i), \mu_B(x_i)) \geq 0 \forall A, B \in X$ for $i = 1, \ldots, n$.
(ii) $d(\mu_A(x_i), \mu_B(x_i)) = 0\ iff\ \mu_A(x_i) = \mu_B(x_i)$ for $i = 1, \ldots, n$.
(iii) $d(\mu_A(x_i), \mu_B(x_i)) = d(\mu_B(x_i), \mu_A(x_i)) \forall A, B \in X$ for $i = 1, \ldots, n$.
(iv) $d(\mu_A(x_i), \mu_B(x_i)) \geq d(\mu_A(x_i), \mu_C(x_i)) + d(\mu_C(x_i), \mu_B(x_i)) \forall A, B, C \in X$ for $i = 1, \ldots, n$.

In words, the definition states that: (a) distances are nonnegative, and the only point at zero distance from $\mu_A(x_i)$ is $\mu_A(x_i)$ itself (b) distance is a symmetric function (c) distances satisfy triangle inequality.

Based on the definition above, we proposed the following metric on fuzzy set A and B drawn from $X = \{x_1, \ldots, x_n\}$ in accordance to distance measures on intuitionistic fuzzy sets proposed by Szmidt [6].

The Hamming distance: $d(\mu_A(x_i), \mu_B(x_i)) = \frac{1}{2}\sum_{i=1}^{n} \left| \mu_A(x_i) - \mu_B(x_i) \right|$ (3)

The Euclidean distance: $d(\mu_A(x_i), \mu_B(x_i)) = \sqrt{\frac{1}{2}\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}$ (4)

The normalized Hamming distance:

$$d(\mu_A(x_i), \mu_B(x_i)) = \frac{1}{2n}\sum_{i=1}^{n} \left| \mu_A(x_i) - \mu_B(x_i) \right| \quad (5)$$

The normalized Euclidean distance:

$$d(\mu_A(x_i), \mu_B(x_i)) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2} \quad (6)$$

(3)=(5) and (4)=(6) whenever $i = n$.

Given that $A = \{0.7, 0.4\}$ and $B = \{0.5, 0.8\}$ for $X = \{x_1, x_2\}$. Using (3) to (6), (3) yields 0.3, (4) yields 0.316, (5) yields 0.15 and (6) yields 0.22.

From this experiment, (5) gives the best metric, follow by (6), (3) and (4).

Theorem 3. Let X be nonempty and $A, B \in X$ be fuzzy sets, then $d(\mu_A(x_i), \mu_B(x_i)) = d(\mu_{A'}(x_i), \mu_{B'}(x_i))$ for $i = 1, \ldots, n$.

Proof. We prove that $d(\mu_A(x_i), \mu_B(x_i)) = d(\mu_{A'}(x_i), \mu_{B'}(x_i))$. But $\mu_{A'}(x_i) = 1 - \mu_A(x_i)$ and $\mu_{B'}(x_i) = 1 - \mu_B(x_i)$. Using the Hamming metric, $d(\mu_A(x_i), \mu_B(x_i)) = \frac{1}{2}\sum_{i=1}^{n} \left|\mu_A(x_i) - \mu_B(x_i)\right|$. If we show that $d(\mu_{A'}(x_i), \mu_{B'}(x_i)) = \frac{1}{2}\sum_{i=1}^{n} \left|\mu_A(x_i) - \mu_B(x_i)\right|$, we are done.

Now, $d(\mu_{A'}(x_i), \mu_{B'}(x_i)) = \frac{1}{2}\sum_{i=1}^{n} \left| 1 - \mu_A(x_i) - 1 + \mu_B(x_i) \right| = \frac{1}{2}\sum_{i=1}^{n} \left| -(\mu_A(x_i) - \mu_B(x_i)) \right| = \frac{1}{2}\sum_{i=1}^{n} \left| \mu_A(x_i) - \mu_B(x_i) \right|$ as desired.□

Corollary 1. If $d(\mu_A(x_i), \mu_B(x_i))$ is a metric of fuzzy sets A and B, then $d^*(\mu_A(x_i), \mu_B(x_i)) = \frac{1}{2}[d(\mu_A(x_i), \mu_B(x_i)) + d(\mu_B(x_i), \mu_A(x_i))]$ is also a metric of A and B, for $i = 1, \dots, n$.

Proof. Our task is to show that $d^*(\mu_A(x_i), \mu_B(x_i)) = \frac{1}{2}[d(\mu_A(x_i), \mu_B(x_i)) + d(\mu_B(x_i), \mu_A(x_i))]$ is a metric. From the definition of metric, $d(\mu_A(x_i), \mu_B(x_i)) = d(\mu_B(x_i), \mu_A(x_i))$ and so, $d^*(\mu_A(x_i), \mu_B(x_i))$ is also a metric.□

Lemma 2. If $d(\mu_A(x_i), \mu_B(x_i))$ is a metric of fuzzy sets A and B, then $d(\mu_A(x_i), \mu_B(x_i)) - d(\mu_B(x_i), \mu_A(x_i)) = 0$ for $i = 1, \dots, n$.

Proof. The proof is directly from the definition of metric, since $d(\mu_A(x_i), \mu_B(x_i)) = d(\mu_B(x_i), \mu_A(x_i)) \Rightarrow d(\mu_A(x_i), \mu_B(x_i)) - d(\mu_B(x_i), \mu_A(x_i)) = 0$ for $i = 1, \dots, n$.□

5. Conclusion

Since the membership degree of a fuzzy set is a function, the concept of norm and metric is very interesting in the context of fuzzy set because fuzzy set is an extension of Cantor's set (note that, norm and metric are defined on set).

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