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Stable Topology for 0-distributive lattices

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Abstract. In this paper, we introduce and study the stable topology on the set of prime filters of a bounded 0-distributive lattice. The stable topology is a subtopology of the hull kernel topology on the set of prime filters of a bounded 0-distributive lattice. Sufficient condition is given under which the hull kernel topology and stable topology coincide on the set of prime filters (the set of maximal filters and the set of minimal prime filters) of a bounded 0-distributive lattice.

Keywords: Maximal filters, minimal filters, normal lattice, dually stone lattice

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1. Introduction and preliminaries

As a generalization of a distributive lattice (with 0) on one hand and a pseudo complemented lattice on the other, Varlet [10] has introduced the concept of a 0-distributive lattice. Dually we define 1-distributive lattice. Various properties of 0-distributive lattices are obtained in [6,7,9,10]. An extensive study of the hull kernel topology on the set of prime filters of a 0-distributive lattice is carried out in [5,6]. In this paper, exactly as in BL-algebras, we introduce the stable topology on the set of prime filters of a 0-distributive lattice ([1,6]). We show that the stable topology is a subtopology of hull kernel topology. It is observed that under certain conditions these two topologies coincide on the set of prime filters, maximal filters and minimal prime filters of a0-distributive lattice. For the topological concepts which have now become commonplace the reader is referred to [4]. For the lattice theoretic concepts the reader is referred to [3].

A lattice L with 0 is 0-distributive, if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$ for $a, b, c \in L$. A lattice L with 1 is 1-distributive, if $a \vee b = 1$ and $a \vee c = 1$ imply $a \vee (b \wedge c) = 1$ for $a, b, c \in L$. A bounded lattice which is both 0-distributive and 1- distributive is called 0-1 distributive lattice [5]. An element $x \in L$ has a complement in L if there exists $x' \in$ L such that $x \wedge x' = 0$ and $x \vee x' = 1$. A non empty subset F of lattice L is called as filter if it is sublattice of L and $x \in F, y \in L$ imply that $x \vee y \in F$. A filter P in a bounded lattice L is said to be prime if P $\neq$ L and $x \vee y \in P$ imply that $x \in P$ or $y \in P$. A filter P in a bounded lattice L is a minimal prime filter if P is a prime filter in L and P does not contain any other prime filter properly. A filter F in L is said to be maximal if there does not exist any proper filter containing it properly. Now onwards L denotes a bounded 0-distributive lattice, F(L) denotes the set of all filters of L and Speck L the set of all prime filters of L. We know that $\langle$ F(L), $\wedge$, $\vee$ $\rangle$ is a bounded lattice where $F \vee K = [F \cup K)$ and $F \wedge K = F \cap K$ for $F, K \in F(L)$.

2. Stable topology

For any filter F of L, U(F)={ P $\in$ Speck L/ F $\subseteq$ P} and D(F) =Speck L- U(F) = { P $\in$ Speck L/ F $\nsubseteq$ P}. Since L is 0-distributive every proper filter of L is contained in a prime filter (see [10]). Hence U (F) is not empty, also D(F) is not empty if $F \neq L$.

We need the following result for the development of the text.

Theorem 2.1. (see [6]) Let $\{A_i / i \in I\}$ be any family of filters of *L* and *A*$_1$,…..,*A*$_n$ be any finite number of filters of *L*. Then

1. $D(\vee A_i) = \cup D(A_i)$
2. $D(A_1 \cap \ldots \cap A_n) = D(A_1) \cap \ldots \cap D(A_n)$
3. $D(L) = p$
4. $D([1]) = \phi$

By Theorem 2.1, it follows that $\{D(F)/F \in F(L)\}$ is a topology on Speck L. We shall denote this topology by $\mathcal{J}$ and the resulting topological space (*Speck L*, $\mathcal{J}$) also by Speck L when there is no ambiguity. This topology is the hull kernel topology. The family $\mathcal{B} = \{ D(x)/x \in L\}$ forms a base for the topology $\mathcal{J}$, where D(x) = D([x)) = $\{P \in$ Speck L$/\ x \notin P\}$.(see [6])

Now we define

Definition 2.1. A subset X of Speck L is called stable if for any P, Q $\in$ Speck L, whenever P $\subseteq Q$ and P $\in$ X, Q $\in$ X.

Using stable open sets in Speck L, We have

Theorem 2.2. Let $\mathcal{J}'$= {D(F) / F is filter in L and D(F) is stable in Speck L}.Then $\mathcal{J}'$is topology on Speck L.

Proof - (i) φ $\in \mathcal{J}'$and Speck L $\in \mathcal{J}'$since φ =D([1)) and Speck L =D(L).

(ii) Let D(F$_1$) and $\mathrm{D(F_2)} \in \mathcal{J}'$. As $\mathrm{D(F_1) \cap D(F_2) = D(\ [F_1 \cup F_2)\ )}$ (by Theorem 2.1), we only prove that $\mathrm{D(F_1) \cap D(F_2)}$ is stable in Speck L. Let $\mathrm{P} \in \mathrm{D(F_1) \cap D(F_2)}$, P , Q $\in$ Speck L and P $\subseteq Q$. P $\in \mathrm{D(F_1)}$ and D(F$_1$) is stable in Speck L will imply Q $\in \mathrm{D(F_1)}$. Similarly, P $\in \mathrm{D(F_2)}$ and D (F$_2$) is stable in Speck L will imply Q $\in \mathrm{D(F_2)}$. Therefore Q $\in \mathrm{D(F_1) \cap D(F_2)}$. Hence $\mathrm{D(F_1) \cap D(F_2)}$ is stable. Therefore $\mathrm{D(F_1) \cap D(F_2)} \in \mathcal{J}^{'}$. (iii) Let$\{\mathrm{D\ (F_\alpha)}\}_{\alpha\in\Delta}$ be any arbitrary family in $\mathcal{J}'$. Then to prove $\bigcup\limits_{\alpha\in\Delta} D(\mathrm{F}_\alpha) \in \mathcal{J}'$. We know $\bigcup\limits_{\alpha\in\Delta} D(\mathrm{F}_\alpha) = D\left(\bigcap\limits_{\alpha\in\Delta} \mathrm{F}_\alpha\right)$(by Theorem 2.1). Hence we only prove that $\bigcup\limits_{\alpha\in\Delta} D(\mathrm{F}_\alpha)$ is stable inSpeck L. Let P, Q $\in$ *Speck L*, P $\subseteq Q$ and P $\in \bigcup\limits_{\alpha\in\Delta} D(\mathrm{F}_\alpha)$. Then P $\in$ D (F_{α_0}) for some $\alpha_0 \in \Delta$. AsD (F_{α_0}) $\in \mathcal{J}'$and as it is stable in Speck L. Q $\in$ D (F_{α_0}). But then we get Q $\in$ D (F_{α_0}). This implies that $\bigcup\limits_{\alpha\in\Delta}$ D(F_α) $\in \mathcal{J}'$. Hence from (i), (ii) and (iii) we get $\mathcal{J}'$ is topology on Speck L. □

As for any filter F, $\mathrm{D(F)} \in \mathcal{J}$, we get

Definition 2.2. This topology $\mathcal{J}'$defined on Speck L is called the stable topology on Speck L

Corollary 2.1. $\mathcal{J}'$ is subtopology of $\mathcal{J}$ on the Speck L i.e. $\mathcal{J}' \subseteq \mathcal{J}$.

Theorem 2.3. $(Speck\ L\ , \mathcal{J}')$ is compact but not T_0.
Proof . As $\mathcal{J}' \subseteq \mathcal{J}$ and $\mathcal{J}$ is compact (see [6]), it follows that $\mathcal{J}'$ is compact. Let $\mathrm{P}, \mathrm{Q} \in Speck\ L$ with $\mathrm{P} \subset \mathrm{Q}$ .Then for any open set D (F) in $\mathcal{J}'$ containing P will surely contain Q as D (F) is stable. This shows that such $\mathrm{P}, \mathrm{Q} \in Speck\ L$cannotbe separated by any open set in $\mathcal{J}'$. Hence $\mathcal{J}'$ is not a T_0 Space.□

Remark 2.1. As we know $\mathcal{J}'$ is a T_1-space, the two topologies $\mathcal{J}$ and $\mathcal{J}'$ defined on Speck L are different.
A sufficient condition for the basic open set D(x) in $\mathcal{J}$ is given in the following theorem.

Theorem 2.4. If $x \in L$ has a complement in L, then D(x) is stable.
Proof. Let $\mathrm{P}, \mathrm{Q} \in Speck\ L$, $P \subseteq Q$ and $P \in \mathrm{D}(\ x)$. By assumption x has a complement say $x' \in$ L. Hence $x \wedge x' = 0$ and $x \vee x' = 1$. Now $P \in \mathrm{D}(x) \Longrightarrow x \notin P \Longrightarrow x' \in$ P. Let $P \subseteq Q$and $x' \in$ P $\Longrightarrow x' \in$ Q $\Longrightarrow x \notin$ Q $(\because x \wedge x' = 0)$. Hence $Q \in \mathrm{D}(x)$. Hence D(x) is stable.□

Theorem 2.5. The stable topology and hull kernel topology coincides on Speck L when L is complemented.
Proof. Let F be a filter in L. We know that $\mathrm{D(F)} = \bigcup_{x \in F} \mathrm{D}(x)$. As $x \in L$ has a complement in L, D(x) is stable (by Theorem2.4). Arbitrary union of stable open sets being stable open (Theorem 2.1), D(F) is stable. Thus any open set in $\mathcal{J}$ is open in $\mathcal{J}'$. As $\mathcal{J}' \subseteq \mathcal{J}$ always We get $\mathcal{J} = \mathcal{J}'$. □

Theorem 2.6. Let L be a 0-1 distributive lattice Then D (F) is stable if the filter F satisfies the following condition..
(*) for every $f \in \mathrm{F}, \exists\ x \in$ L and $y \in$ F such that $x \vee f = 1$ and $x \wedge y = 0$.
Proof. Let $\mathrm{P}, \mathrm{Q} \in Speck\ L$ such that $P \subseteq Q$ and $\mathrm{P} \in \mathrm{D}$ (F). So F $\nsubseteq$ P. Hence $\exists\ f \in$ F such that $f \notin$ P. By the property of F, there exist $x \in L$ and $y \in F$ such that $x \wedge y = 0$ and $x \vee f = 1$. Now $x \vee f = 1 \in$ Pand P is prime will give $x \in \mathrm{P}(\because f \notin \mathrm{P})$. Thus $x \in$ P $\subseteq$ Q. As $x \wedge y = 0$ and∈ Q, we get $y \notin Q$, since Q is proper. Thus $y \in$ F and $y \notin$ Q imply F $\nsubseteq$ Q. But this gives $\mathrm{Q} \in \mathrm{D(F)}$ and hence D (F) is stable.□

In a 0-distributive lattice we know Max L is subset of Speck L. Hence by restricting the hull kernel topology $\mathcal{J}$ defined on Speck L to the set Max L we obtain the relative topology say $\mathcal{J}_1$on Max L. Restrict the stable topology $\mathcal{J}'$defined on the Speck L to Max L and denote it by $\mathcal{J}_1'$.Thus $\mathcal{J}_1$denotes the hull-kernel topology on Max L and$\mathcal{J}_1'$ denote the stable topology on Max L.

A bounded 0-distributive lattice is normal if every prime filter in L is contained in a unique maximal filter. Under the condition of normality of L we get $\mathcal{J} = \mathcal{J}'$. This we prove in the following theorem.

Theorem 2.7. If L is a normal lattice, then the hull kernel topology coincides with the stable topology on Max (L).

Proof. For any $M \in \text{ Max (L)}$, let $D^M(x) = \text{D (} x \text{) } \cap \text{Max(L)}$ for each $x \in$ L. Then for each $x \in$ L,$\mathrm{D^M}(x)$ is a basic open set in $(\text{Max(L)}, \mathcal{J}_1)$. L being normal, every prime filter P in L is contained in a unique maximal filter, say$\mathrm{M_p}$. Define $\mathrm{V} = \{\mathrm{P \in Speck\ L/\ M_p \in D^M}(x)\}$.Therefore$\mathrm{V} = \{\mathrm{P \in Speck\ L/}\ x \notin \mathrm{M_p}\ {}_{,}\mathrm{M_p} \in \mathrm{Max(L)}\}$.Then $\mathrm{D^M}(x) = \mathrm{V} \cap \text{ Max(L)}$. To prove $\mathrm{D^M}(x)$ is stable, it is enough to prove V is stable. Let $\mathrm{P, Q} \in Speck\ L, P \subseteq Q$ and $P \in$ V.Then $\mathrm{M_P = M_Q}$and$P \in$ Vgives $x \notin \mathrm{M_Q}$. Hence $\mathrm{Q \in V}$. This shows that V is stable and hence $\mathrm{D^M}(x) = \mathrm{V} \cap \text{ Max (L)}$ is stable. This in turn shows that $\mathcal{J}_1 \subseteq \mathcal{J}_1{}'$,As $\mathcal{J}_1{}' \subseteq \mathcal{J}_1$(see Corollary 2.1) we get $\mathcal{J}_1 = \mathcal{J}_1{}'$ i.e. the hull kernel topology coincides with the stable topology on the set of all maximal filters in L.□

A 0-1 distributive lattice L is called Stone lattice if $(x]^* = (y]$ for some complemented element y in L. A 0-1 distributive lattice L is called dually Stone lattice if $[x)^{\perp} = [y)$ for some complemented element y in L.

Theorem 2.8. Let L be a 0-1 distributive lattice. If L is a dually stone lattice, then for any $\in$ L , there exists$y \in$ Lsuch that y has a complement say y' in L and $\mathrm{D_M}(x) = \mathrm{D_M}(y')$.

Proof. Let $x \in$ L , As L is a dually stone lattice, there exists a complemented element $y \in$ L such that $[x)^{\perp} = [y)$. Then $x \vee y = 1, y \wedge y' = 0$ and $y \vee y' = 1$. $\mathrm{P} \in \mathrm{D_M}(x)$ implies $x \notin P$ and $\mathrm{P \in M}$. Therefore $y \in$ P (since $x \vee y = 1 \in$ P, P is prime). Thus $y' \notin P$ (as $y \wedge y' = 0$, P is prime) implies $\mathrm{P} \in \mathrm{D_M}(y')$. Hence $\mathrm{D_M}(x) \subseteq \mathrm{D_M}(y')$. To prove the reverse inclusion, letP $\in \mathrm{D_M}(y')$. Then $y' \notin P$.Therefore $y \in$ P (since $y \vee y' = 1$) that is $[y) \subseteq$ Pimplies$[x)^{\perp} \subseteq$ P. As $P \in$ M, we get $x \notin P$ and consequently $\mathrm{P} \in \mathrm{D_M}(x)$.Thus$\mathrm{D_M}(y') \subseteq \mathrm{D_M}(x)$. Combining both the inclusions we get $\mathrm{D_M}(x) = \mathrm{D_M}(y')$.□

Min (L) is subset of Speck L. Hence by restricting the hull kernel topology $\mathcal{J}$ defined on Speck L to the set Min (L) we obtain the relative topology say $\mathcal{J}_2$, on Min(L).Restrict the stable topology on Min L and denote it by $\mathcal{J}_2{}'$. Interestingly we have,

Theorem 2.9. The hull kernel topology and the stable topology coincide on the set of all minimal prime filters of 0-1 distributive dually Stone lattice.

Proof. Let L be a stone lattice. Let $x \in$ L and $\mathrm{D_M}(x) = D(x) \cap Min(L)$. Then $\{D_M\ (x)/x \in \mathrm{L}\}$ forms a base for $\mathcal{J}_2$. Since L is dually stone lattice there exists a complemented element y in L such that $[x)^{\perp} = [y)$. Now (by theorem 2.8) $\mathrm{D_M}(x) =$

$\mathrm{D_M}(y')$. As $\mathrm{D_M}(y')$ is stable in Speck L (by theorem 2.4), we get $\mathrm{D_M}(x)$ is stable for every$x \in$ L. This shows that $\mathcal{J}_2$ and stable topology coincides on Min (L). $\square$

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