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Estimation of Stationary Distribution with Application to ABO Blood Group using Spectral Expansion Theory

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Abstract. Every human being can be classified into one of the four blood groups such as A, B, AB, O. Researchers have drawn much attention in determining the risk factors involved in the blood group among the mothers and newly born infants. Letsky et al., have analyzed the rhesus and other haemolytic diseases. Taramoon has examined the relationship of ABO blood group and life span in a hospitalized population in South eastern united states. In this article we have used the concept of spectral expansion theory to determine the strong relationship among mother's and matured neonates blood group.

Keywords: ABO blood group, spectral expansion, transition probability, stationary distribution.

AMS Mathematics Subject Classification (2010): 60J05, 60J10, 60J20, 65C50, 65C60

1. Introduction

A blood type is a classification of blood based in the presence or absence of inherited antigenic substances on the surface of red blood cells (RBCs). These antigens may be proteins, carbohydrates, glycoproteins or glycolipids depending on the blood group system. Some of these antigens are also present on the surface of other types of cells of various tissues. Blood types are inherited and represent contributions from both parents. The most important human blood group are ABO and the RbD antigen where the ABO blood group antigens remain of prime importance in transfusion medicine because they are the most immunogenic of all the blood group antigens.

Blood group AB individuals have both A and B antigens on the surface of their RBCs and their plasma does not contain any antibodies against either A or B antigen. Hence an individual with AB blood group can receive blood from any group but cannot donate blood to any group other than AB

Blood group A individuals have the A antigen in the surface of their RBCs and blood serum containing Igm antibodies against the B antigen. Hence the group A individual can receive blood only from individuals of blood group A or O and can donate blood to individuals with type A or AB.

Blood group B individual have the B antigen on the surface of their RBCs and blood serum contains Igm antibodies against the A antigens. Therefore a group B individual can receive blood only from group B or O and can donate blood to B or AB type individual

Blood group O do not have either A or B antigens but contains IgM and anti -A and anti-B antibodies against A and B blood group antigens. Hence group O individual can receive blood only from O but can donate to individual of any ABO blood group.

Many pregnant women carry a fetus with a blood type which is different from their own and the mother can form antibodies against fetal RBCs. During gestational period the mother and baby blood generally do not mix. Their blood circulation is kept separate by the placental membrane. Sometimes these materials antibodies are a small immunoglobulin which can cross the placenta and cause hemolysis which in turn can lead to hemolytic disease of the newborn. In various research it has been found that premature babies experience severe problems from ABO incompatibility while healthy full term babies are generally only mildly affected. Various options are available for testing ABO incompatibility which is far behind the scope of this study.

In this article we have collected the samples of both mothers and babies blood group of matured live births from various hospital and spectral expansion theory concept has been used to determine the equilibrium vector for the relationship between their blood group.

2. Preliminaries

Let $\{X(t), t = 0,1, ...\}$ be a Markov chain with state space S={0,1,2,….} . The transition probabilities are given by $p_{ij}(m,n) = \Pr\{X(n) = j/X(m) = i\}$. As we know one of the basic approach to prove results in Markov chain is the well known equation called as the Chapman Kolmogrov equation which states that "Let $\{X(t), t\varepsilon T\}$ be a Markov chain with state space $S = \{0,1,2, ...\}$ and parameter space $T = \{0,1,2, ...\}$. The transition probability is given by $p_{ij}(m,n) = P\{X(n) = j/X(m) = i\}$.

Then, for any 0<m<r<n, we have $p_{ij}(m,n) = \sum_{k=0}^{\infty} p_{ik}(m,r)p_{kj}(r,n)$

The above equation is the basic approach for proving many of the results in Markov chains.

When the chain is finite,

$$P(m,n) = (p_{ij}(m,n)) = P(m,r)P(r,n)$$
$$= \prod_{r=1}^{n-m} P(m+r-1, m+r)$$

When the chain is finite and homogeneous then

$$P(n, n+1) = P$$

which is independent of n so that $P(m,n) = P^{n-m}$

This can also be extended to some non-homogeneous finite Markov chains. Suppose

$$p_{ij}(n, n+1) = p_{ij}(s) \text{ for } t_{s-1} \leq n \leq t \text{ , } s=1,2,\ldots N$$

then with t_0=0,

$$P(0,n) = P_1^n \; if \; n \leq t_1$$

$$= \left\{\prod_{i=0}^{s-1} P_{i+1}^{t_{s+1}-t_s}\right\} P_{s+1}^{n-t} \quad if\ t_1 < t_2 < \cdots t_s \le n < t_{s+1}\ where\ s = 1,2, \dots N-1$$

From this it follows P(0,n) has to be computed for finite Markov chains with discrete time where as in many cases it is necessary to find the power of one step transition matrices.

3. Model description

The blood group of mothers and babies of matured live births were taken from 400 cases and are shown in the table 1.

Mothers blood group	Babies blood group			
	A	*B*	*AB*	*O*
A	14	7	18	8
B	29	48	90	34
AB	7	11	39	14
O	47	14	0	20

Table 1**:** Blood group of mothers and babies

The mother-baby relation among the matured live births is determined by the transition robability matrix *P* as shown in table 2.

Birth of babies

Mothers blood group	Babies blood group			
	A	*B*	*AB*	*O*
A	0.2979	0.1489	0.3830	0.1702
B	0.1442	0.2388	0.4478	0.1692
AB	0.0986	0.1549	0.5493	0.1972
O	0.5802	0.1728	0	0.2469

Table 2**:** Transition probability matrix of mothers and matured live

Consider a $n \times n$matrix P whose eigen values of P must satisfy $\left| P - \lambda I_n \right| = 0$ which is a polynomial in λ of degree n which has n roots in the complex field.

Let the distinct roots of P be $\lambda_1, \lambda_2, \lambda_3 \dots, \lambda_r$ repeated $k_1, k_2, k_3 \dots, k_r$ times respectively where $k_1 + k_2 + k_3 + \cdots + k_r = n$.

The matrix P is diagonable iff there exists a non-singular $n \times n$ matrix S such that

$$S^{-1}PS = P = \begin{pmatrix} \lambda_1 I_{k_1} & \dots\dots & 0 \\ \vdots & \lambda_2 I_{k_2} & \vdots \\ 0 & \dots & \lambda_r I_{k_r} \end{pmatrix}$$

is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \lambda_3 \dots, \lambda_r$ repeated $k_1, k_2, k_3 \dots, k_r$ times respectively.

Partitioning S as $S = (S_1, S_2, \dots S_r)$ and $S^{-1}\ by\ S^{-1} = \left(T_1^{'}, T_2^{'}, \dots T_r^{'}\right)^{'}$

$$S^{-1}PS = P$$

$$\Rightarrow PS = SP, S^{-1}P = PS^{-1}$$

$$\Rightarrow P = SPS^{-1}$$

Hence $PS = SP \Rightarrow PSi = \lambda iSi$, i=1,2,…r
and $S^{-1}P = PS^{-1} \Rightarrow T_iP = \lambda_i T_i, i = 1,2, \dots r$
Here the columns of Si are independent right eigen vectors of the eigen value $\lambda_i$and rows of T_i are independent left eigen vectors of λ_i.
Also

$$P = SPS^{-1} = (S_1, S_2, \dots S_r)P\left(T_1', T_2', \dots T_r'\right)'$$

$$= \sum_{i=1}^{r} \lambda_i S_i T_i$$

$$= \sum_{i=1}^{r} \lambda_i E_i \ where\ E_i = S_i T_i\ , i = 1,2, \dots r$$

$$Now \sum_{i=1}^{r} E_i = \sum_{i=1}^{r} S_i T_i$$

$$= (S_1, S_2, \dots S_r)\left(T_1', T_2', \dots T_r'\right)'$$

$$= SS^{-1} = I_n$$

$$Also\ I_n = S^{-1}S = \left(T_1', T_2', \dots T_r'\right)'(S_1, S_2, \dots S_r)$$

$$= \begin{pmatrix} T_1S_1 & T_1S_2 & & T_1S_r \\ T_2S_1 & T_2S_2 & & T_2S_r \\ & \vdots & & \\ T_rS_1 & T_rS_2 & & T_rS_r \end{pmatrix}$$

$$\Rightarrow TiSi = I_{k_i},\ \ T_iS_j = 0\ \ for\ i \neq j.$$

Thus $E_i^2 = E_iE_i = S_iT_iS_iT_i$
$=S_iT_i = E_i$
Hence $E_i^2 = E_i$ and $E_iE_j = S_iT_iS_jT_j = 0\ for\ i \neq j$
Since the matrix is diagonable, by spectral expansion

$$P = \sum_{j=1}^{r} \lambda_j E_j$$

where $\lambda_1, \lambda_2, \lambda_3 \dots, \lambda_r$ are the distinct eigen values of P and

$$E_j = \prod_{j \neq i} \frac{1}{\left(\lambda_j - \lambda_i\right)} (P - \lambda_j I_n)\, , i = 1,2, \dots r$$

The stationary distribution of the matrix P can be determined using the expression

$$P(n) = P^n = \sum_{j=1}^{r} \lambda_j^n E_j$$

In the case when only some of the eigen values are repeated it is enough to compute E_j only for the distinct eigen values.
When all the eigen values are equal, the stationary distribution can be determined using

$$\lim_{n\to\infty} P^n = (I_r - P)^{-1} = \sum_{i=1}^{r} \frac{1}{1-\lambda_i} E_i \ \ where\ \ 1-\lambda_i > 0$$

The eigen values of the transition probability matrix P is given as

$\lambda_1 = 1,\ \lambda_2 = 0.1185\ +\ 0.0884i,\ \lambda_3 = 0.1185\ -\ 0.0884i, \lambda_4 = 0.0958$

We can notice that all the eigen values are distinct.

Consider $E_1 = \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}(P-\lambda_2 I_4)(P-\lambda_3 I_4)(P-\lambda_4 I_4)$

$$= \begin{bmatrix} 0.2504 & 0.1713 & 0.3830 & 0.1953 \\ 0.2504 & 0.1713 & 0.3830 & 0.1953 \\ 0.2504 & 0.1713 & 0.3830 & 0.1953 \\ 0.2504 & 0.1712 & 0.3829 & 0.1953 \end{bmatrix}$$

$$E_2 = \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}(P-\lambda_1 I_4)(P-\lambda_3 I_4)(P-\lambda_4 I_4)$$

$$= \begin{bmatrix} 0.2870 + 0.2112i & -0.0225 + 0.0279i & -0.1806 - 0.2541i & -0.0850 + 0.0146i \\ 0.5339 + 0.6021i & -0.0626 + 0.0510i & -0.2782 - 0.6457i & -0.1942 - 0.0078i \\ -0.1490 + 0.6846i & -0.0682 - 0.0178i & 0.3105 - 0.5268i & -0.0944 - 0.1406i \\ -0.5452 - 2.1412i & 0.2165 - 0.0460i & -0.1354 + 1.9235i & 0.4632 + 0.2633i \end{bmatrix}$$

$$E_3 = \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}(P-\lambda_1 I_4)(P-\lambda_2 I_4)(P-\lambda_4 I_4)$$

$$= \begin{bmatrix} 0.2870 - 0.2112i & -0.0225 - 0.0279i & -0.1806 + 0.2541i & -0.0850 - 0.0146i \\ 0.5339 - 0.6021i & -0.0626 - 0.0510i & -0.2782 + 0.6457i & -0.1942 + 0.0078i \\ -0.1490 - 0.6846i & -0.0682 + 0.0178i & 0.3105 + 0.5268i & -0.0944 + 0.1406i \\ -0.5452 + 2.1412i & 0.2165 + 0.0460i & -0.1354 - 1.9235i & 0.4632 - 0.2633i \end{bmatrix}$$

$$E_4 = \frac{1}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}(P-\lambda_1 I_4)(P-\lambda_2 I_4)(P-\lambda_3 I_4)$$

$$= \begin{bmatrix} 0.1755 & -0.1263 & -0.0219 & -0.0254 \\ -1.3183 & 0.9539 & 0.1734 & 0.1931 \\ 0.0476 & -0.0349 & -0.0040 & -0.0066 \\ 0.8400 & -0.6042 & -0.1121 & -0.1217 \end{bmatrix}$$

Since the matrix is diagonable, by spectral expansion

$$P = \sum_{j=1}^{4} \lambda_j E_j$$

$$P = \begin{bmatrix} 0.2979 & 0.1489 & 0.3830 & 0.1702 \\ 0.1442 & 0.2388 & 0.4478 & 0.1692 \\ 0.0986 & 0.1549 & 0.5493 & 0.1972 \\ 0.5802 & 0.1728 & -0.0000 & 0.2469 \end{bmatrix}$$

The stationary distribution of the matrix P can be determined using the expression

$$P(n) = P^n = \sum_{j=1}^{r} \lambda_j^n E_j$$

Hence $\lim_{n\to 7} P^n = \begin{bmatrix} 0.2504 & 0.1712 & 0.3829 & 0.1953 \\ 0.2504 & 0.1712 & 0.3829 & 0.1953 \\ 0.2504 & 0.1712 & 0.3829 & 0.1953 \\ 0.2504 & 0.1712 & 0.3829 & 0.1953 \end{bmatrix}$

The equilibrium vector is equal to $(0.2504,\ 0.1712,\ 0.3829,\ 0.1953)$ which shows that the blood group of mothers A, B, AB, O have probability of $0.2504,\ 0.1712,\ 0.3829,\ 0.1953$ with the blood group of babies respectively. Hence we can conclude that mothers with AB blood group have higher probability with the same blood group of babies among matured live births..

To investigate whether there is any significant difference between mothers and neonates blood group, we test the assumptions for the blood group using Chi-square test. We first formulate a conservative hypothesis, called the null hypothesis (H_o), which states that there is no difference between the observed and expected values. The observed frequency and the expected frequency are attained as indicated in the table 3.

Blood group	**Observed Frequency**	**Expected Frequency**
A	97	100.2
B	80	68.5
AB	147	153.2
O	76	78.1

Table 3**:** Comparison of expected and observed frequency

ψ^2 for goodness of fit is given by $\psi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i}$ = 2.3402.

The object is to establish the significant difference for three degrees of freedom at 5% level. The ψ^2 value reveals that there is no significant difference between mothers and neonates blood group for matured births.

4. Conclusion

In this study the results are based on the records collected from the various hospitals. It has been interpreted that the blood group of mothers with AB blood group has highest probability of matured live birth compared to other blood groups namely A, B, O. These findings will be helpful for planning and mobilizing the health care among pregnant women.

REFERENCES

1. Tan Wai-Yuan, *Stochastic Models with Application to Genetics, Cancers, Aids and Other Biomedical System*, World Scientific Co. Pte. Ltd, Singapore (2002).
2. D.Nagarajan and P.Sunitha, Ranking for reordering documents using latent semantic indexing, *International Conference on Advances in Modeling, Optimization and Computing*, IIT Rourkee, (2011) 1074-1077.
3. Tara Moon, An examination of the relationship of ABO blood group and life span in a hospitalized population in South eastern united states Theses and Dissertations, Virginia Commonwealth University (2014).
4. E.A.Letsky, I.Leck, J.M.Bowman, Chapter 12: Rhesus and other haemolytic diseases, *Antenatal & Neonatal Screening* (2nd ed.), Oxford University Press (2000).
5. J.Medhi, *Stochastic process*, First reprint, New Age International Private Ltd. (1996).
6. F.Bernstein, Zusammenfassende Betrachtungen uber die erblichen Blutstrukturen des Menschen, *Z. Indukt. Abstammungs.Vererbungsl.*, 37 (1925) 237-370.
7. S.N.Bernstein, Solution of a mathematical problem connected with the theory of heredity, *Ann. Sci. de l'Ukraine*, 1 (1924) 83- 114.
8. N.Ganikhodjaev, J.I.Daoud and M.Usmanova, Stochastic models of heredity rhesus factor, *Australian Journal of Basic and Applied Sciences*, 4(8) (2010) 3306-3310.
9. Md. Nazmul Hasan, H.Das and P.K.Ghosh, S-I epidemic model incorporating social-economic and cultural causes and effects in developing countries, *Annals of Pure and Applied Mathematics*, 6(2) (2014) 115-124.
10. K.Kumar, Type-2 fuzzy set theory in medical diagnosis, *Annals of Pure and Applied Mathematics*, 9(1) (2015) 35-44.