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On Neighborhood Transformation Graphs

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Abstract. Let G=(V, E) be a graph. Let S be the set of all open neighborhood sets of G. Let x, y, z be three variables each taking value + or –. The neighborhood transformation graph NG^{xyz} is the graph having $V \cup S$ as the vertex set and for any two vertices u and v in $V \cup S$, u and v are adjacent in NG^{xyz} if and only if one of the following conditions holds: (i) $u, v \in V$. x = + if $u, v \in N$ where N is an open neighborhood set of G. x = – if $u, v \notin N$ where N is an open neighborhood set of G (ii) $u, v \in S$. y = + if $u \cap v \neq \phi$. y = – if $u \cap v = \phi$. (iii) $u \in V$ and $v \in S$. z = + if $u \in v$. z = – if $u \notin v$. In this paper, we initiate a study of neighborhood transformation graphs. Also characterizations are given for graphs for which (i) NG^{+++} is connected (ii) $NG = NG^{+++}$ and (iii) $N_S(G) = NG^{+++}$.

Keywords: neighborhood graph, middle neighborhood graph, semientire neighborhood graph, entire neighborhood graph, transformation.

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1. Introduction

All graphs considered here are finite, undirected without loops or multiple edges. Any undefined term in this paper may be found in [1].

Let G=(V, E) be a graph with $|V| = p$ vertices and $|E|= q$ edges. For any vertex $u \in V$, the open neighborhood of u is the set $N(u) = \{v \in V: uv \in E\}$. We call $N(u)$ is the open neighborhood set of a vertex u of G. Let $V=\{u_1, u_2, ..., u_p\}$ and let $S = \{N(u_1), N(u_2), ..., N(u_p)\}$ be the set of all open neighborhood sets of G.

The neighborhood graph $N(G)$ of a graph G=(V, E) is the graph with vertex set $V \cup S$ in which two vertices u and v are adjacent if $u \in V$ and v is an open neighborhood set containing u. This concept was introduced by Kulli in [2]. Many other graph valued functions in graph theory were studied, for example, in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and also graph valued functions in domination theory were studied, for example, in [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

The middle neighborhood graph $M_{nd}(G)$ of a graph G is the graph with the vertex set $V \cup S$ in which two vertices u and v are adjacent if $u, v \in S$ and $u \cap v \neq \phi$ or $u \in V$ and v is an open neighborhood set of G containing u. This concept was introduced by Kulli in [28].

The semientire neighborhood graph $N_s(G)$ of G is the graph with the vertex set $V \cup S$ in which two vertices u and v are adjacent if $u, v \in N$, where N is an open neighborhood set of G or $u \in V$ and v is an open neighborhood set of G containing u. This concept was introduced by Kulli in [29].

The entire neighborhood graph $N_e(G)$ of G is the graph with vertex set $V \cup S$, in which two vertices u and v are adjacent if $u, v \in N$ where N is an open neighborhood set of G or $u,v \in S$ and $u \cap v \neq \phi$ or $u \in V$ and v is an open neighborhood set of G containing u. This concept was introduced by Kulli in [30].

Let $\overline{G}$ be the complement of G.

Recently some transformation graphs were studied, for example, in [31, 32, 33, 34, 35, 36]. In this paper, we introduce neighborhood transformation graphs.

2. Neighborhood transformation graphs

Inspired by the definition of the entire neighborhood graph of a graph, we introduce the neighborhood transformation graphs.

Definition 1. Let $G = (V, E)$ be a graph and let S be the set of all open neighborhood sets of G. Let x, y, z be three variables each taking value + and –. The neighborhood transformation graph NG^{xyz} is the graph having $V \cup S$ as the vertex set and for any two vertices u and v in $V \cup S$, u and v are adjacent if and only if one of the following conditions holds:

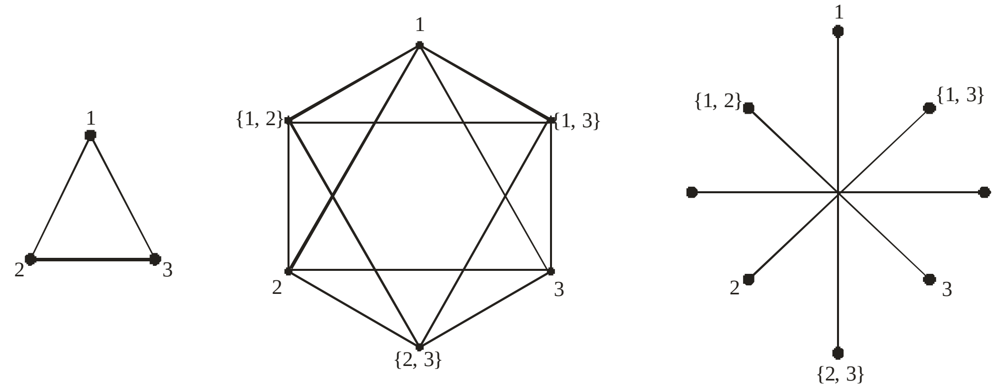
i) $u, v \in V$. $x = +$ if $u, v \in N$ where N is an open neighborhood set of G. $x = -$ if $u, v \notin N$ where N is an open neighborhood set of G.

ii) $u, v \in S$. $y = +$ if $u \cap v \neq \phi$. $y = -$ if $u \cap v = \phi$.

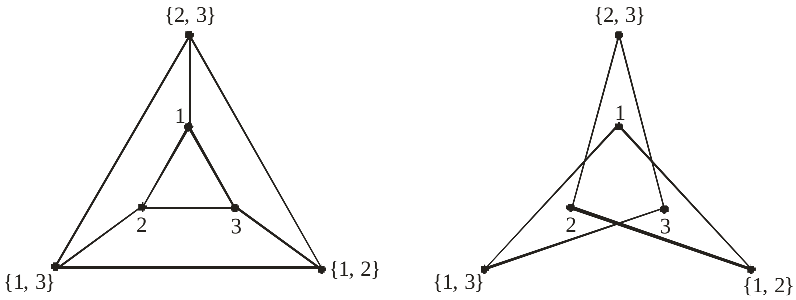
iii) $u \in V$ and $v \in S$. $z = +$ if $u \in v$. $z = -$ if $u \notin v$.

Using the above neighborhood transformation, we obtain eight distinct neighborhood transformation graphs: G^{+++}, G^{+-+}, G^{++-}, G^{-++}, G^{+--}, G^{-+-}, G^{--+}, G^{---}.

Example 2. In Figure 1, a graph G, its neighborhood graphs NG^{+++}, NG^{---}, NG^{++-} and NG^{--+} are shown.



G NG^{+++} NG^{---}



NG^{++-} $\qquad$ NG^{--+}

Figure 1:

Proposition 3. If G is a nontrivial connected graph, then

(1) $\overline{NG^{+++}} = NG^{---}$ $\qquad$ (2) $\overline{NG^{++-}} = NG^{--+}$

(3) $\overline{NG^{+-+}} = NG^{-+-}$ $\qquad$ (4) $\overline{NG^{+--}} = NG^{-++}$

Proof: Each follows from the definitions of G^{+++} and $\overline{G}$.

3. The neighborhood transformation graph NG^{+++}

Among the neighborhood transformation graphs one is the entire neighborhood graph $N_e(G)$. It is easy to see that

Proposition 4. For any graph G, $N_e(G) = NG^{+++}$.

Remark 5. For any graph G, the neighborhood graph $N(G)$ of G is a spanning subgraph of NG^{+++}.

Remark 6. For any graph G, the middle neighborhood graph $M_{nd}(G)$ of G is a spanning subgraph of NG^{+++}.

Remark 7. For any graph G, the semientire neighborhood graph $N_s(G)$ of G is a spanning subgraph of NG^{+++}.

We need the following result to prove our next result.

Theorem A [2]. Let G be a connected graph. The neighborhood graph $N(G)$ of G is connected if and only if G contains an odd cycle.

Theorem 8. Let G be a connected graph. The neighborhood transformation graph NG^{+++} of G is connected if and only if G contains an odd cycle.

Proof: Let G be a connected graph. Suppose G contains an odd cycle. By Theorem A, $N(G)$ is connected. Since, by Remarks 5, $N(G)$ is a spanning subgraph of NG^{+++}, it implies that NG^{+++} is connected.

Conversely suppose NG^{+++} is connected. By Remark 5, $N(G)$ is a spanning subgraph of NG^{+++}. Therefore $N(G)$ is connected. Hence by Theorem A, a connected graph G contains an odd cycle.

Corollary 9. For any nontrivial bipartite graph G, NG^{+++} is disconnected.

Observation 10. If G is a nontrivial connected bipartite graph G, then NG^{+++} has exactly two components.

Theorem 11. $NG^{+++} = 2pK_2$ if and only if $G = pK_2$, $p \geq 1$.
Proof: Suppose $G = pK_2$. Then each open neighborhood set of a vertex of G contains exactly one vertex. Thus the corresponding vertex of open neighborhood set is adjacent with exactly one vertex in NG^{+++}. Since G has $2p$ vertices, it implies that G has $2p$ open neighborhood sets. Thus NG^{+++} has $4p$ vertices and the degree of each vertex is one. Hence $NG^{+++} = 2pK_2$.

Conversely suppose $NG^{+++} = 2pK_2$. We now prove that $G = pK_2$. On the contrary, assume $G \neq pK_2$, Then there exists at least one open neighborhood set containing at least two vertices of G. Then NG^{+++} contains a subgraph P_3. Hence $NG^{+++} \neq 2pK_2$, which is a contradiction. Thus $G = pK_2$.

Theorem 11. $NG^{+++} = 2K_p$ if and only if $G = K_{1,\, p-1}$, $p \geq 2$ or C_p, $p = 4$.
Proof: Let $G = K_{1,\, p-1}$, $p \geq 2$. Let $V(G) = \{v, v_1, v_2, \ldots, v_{p-1}\}$. Let deg $v = p - 1$ and deg $v_i = 1$, $1 \leq i \leq p-1$. Then $N(v) = \{v_1, v_2, \ldots v_{p-1}\}$, $N(v_i) = \{v\}$, $1 \leq i \leq p-1$. Therefore $V(NG^{+++}) = \{v, v_1, v_2, \ldots, v_{p-1}, N(v), N(v_1)\ N(v_2), \ldots, N(V_{p-1})\}$. By Corollary 9, NG^{+++} is disconnected. Since $N(v) = \{v_1, v_2, \ldots, v_{p-1}\}$, it implies that every pair of vertices of $v_1, v_2, \ldots, v_{p-1}$ are adjacent in NG^{+++} and the vertex $N(v)$ is adjacent with every vertex v_i, $1 \leq i \leq p-1$ in NG^{+++}. This produces K_p in NG^{+++}. Since $N(v_i) \cap N(v_j) \neq \phi$, $1 \leq i \leq p-1$, $1 \leq j \leq p-1$, $i \neq j$, it implies that every pair of vertices of $N(v_1), N(v_2), \ldots, N(v_{p-1})$ are adjacent in NG^{+++} and the vertex v is adjacent with the vertices $N(v_1), N(v_2), \ldots, N(v_{p-1})$ in NG^{+++}. This produces K_p in NG^{+++}. Thus the resulting graph is $K_p \cup K_p$. Hence $NG^{+++} = 2K_p$.

Let $G = C_4$. Then it is easy to see that $NG^{+++} = 2K_4$.

Conversely suppose $NG^{+++} = 2K_p$. We prove that G is either $K_{1,\, p-1}$, $p \geq 2$ or C_P, $p = 4$. Since NG^{+++} is disconnected, it implies by Theorem 8, that G has no odd cycles. We consider the following two cases.
Case 1: Suppose G has even cycles and $G \neq C_4$. Then each component of NG^{+++} is not K_p, which is a contradiction. This proves that $G = C_4$.
Case 2. Suppose G is a tree. We now prove that $G = K_{1,\, p-1}$, $p \geq 2$. On the contrary, G is not a star. Then $\Delta(G) < p - 1$. Therefore the open neighborhood set of a vertex of G contains at most p – 2 vertices. Then in each component of NG^{+++}, there exists a vertex whose degree is less than $p - 1$. Thus NG^{+++} does not contain K_p as a component, a contradiction. Thus $G = K_{1,\, p-1}$, $p \geq 2$.

Remark 12. If $G = K_{1,3}$, then $NG^{+++} = 2K_4$. Also if $G = C_4$, then $NG^{+++} = 2K_4$. Clearly $NK_{1,3}^{+++} = NC_4^{+++}$ but $K_{1,3} \neq C_4$.

We characterize graphs G for which $NG^{+++} = N(G)$.

Theorem 13. For any graph G without isolated vertices,
$$N(G) \subseteq NG^{+++}. \qquad (1)$$

Furthermore, equality holds if and only if every open neighborhood set contains exactly one vertex.

Proof: By Remark 5, $N(G) \subseteq NG^{+++}$.

We now prove the second part.

Suppose $NG^{+++} = N(G)$. Assume the open neighborhood set of a vertex of G contains at least two vertices, say $v_1, v_2, \ldots, v_p$, $p \geq 2$. Then the corresponding vertices of $v_1, v_2, \ldots, v_p$ are not mutually adjacent in $N(G)$, but they are mutually adjacent in NG^{+++}. Thus two or more vertices of G are not in the same open neighborhood set.

Conversely suppose every open neighborhood set of G contains exactly one vertex. Then every pair of open neighborhood sets of G is disjoint. Thus the corresponding vertices of open neighborhood sets in NG^{+++} are not adjacent. Hence $NG^{+++} \subseteq N(G)$ and since $N(G) \subseteq \mathrm{NG}^{+++}$, it implies that $N(G) = NG^{+++}$.

We now characterize graphs G for which $NG^{+++} = N_s(G)$.

Theorem 14. For any graph G without isolated vertices,

$$N_s(G) \subseteq NG^{+++}.$$

Furthermore, equality holds if and only if every pair of open neighborhood sets of vertices of G is disjoint.

Proof: By Remark 7, $N_s(G) \subseteq NG^{+++}$.

We now prove the second part.

Suppose $N_s(G) = NG^{+++}$. We prove that every pair of open neighborhood sets of vertices of G is disjoint. On the contrary, assume $N_1, N_2, \ldots, N_k$, $k \geq 2$ are open neighborhood sets of vertices of G such that $N_1 \cap N_2 \neq \phi$. Then the corresponding vertices of N_1 and N_2 are not adjacent in $N_s(G)$ and are adjacent in NG^{+++}. Thus $N_s(G) \neq NG^{+++}$, which is a contradiction. Hence every pair of open neighborhood sets of G is disjoint.

Conversely suppose every pair of open neighborhood sets of G is disjoint. Then two vertices corresponding to open neighborhood sets cannot be adjacent in NG^{+++}. Thus $NG^{+++} \subseteq N_s(G)$ and since $N_s(G) \subseteq NG^{+++}$, it implies that $N_s(G) = NG^{+++}$.

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