Annals of Pure and Applied Mathematics
Vol. 10, No. 2, 2015, 247-254
ISSN: 2279-087X (P), 2279-0888(online)
Published on 17 November 2015
www.researchmathsci.org

Strong Circuit Matrix and Strong Path Matrix of a Semigraph

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Received 13 October 2015; accepted 2 November 2015

Abstract. In this paper the strong circuit matrix and strong path matrix of semigraphs are defined and their relation with partial edge incidence matrix are obtained. The results of circuit matrix and path matrix of simple graph are generalized in this paper.

Keywords: Strong circuit matrix of semigraph; Path matrix of semigraph

AMS Mathematics Subject Classification (2010): 05C50

1. Introduction

The notion of a Semigraph is a new concept introduced by Sampatkumar [5], generalizing the concept of a graph. Semigraph resembles graph when drawn on a plane and every concept/results in graph can be easily generalized yielding a rich variety of corresponding results. Road networks, projective geometry, Steiner's triple systems are the some examples of semigraphs. Many authors [1,2,3,9,10] have studied properties of semigraphs.

Representation of any discrete structure in matrix form is important for the applications in electrical network analysis, operation research and computer science. Many authors [2,7,12,13] have studied the properties graph, semigraph and fuzzy graph by using their associated matrices. The author [7] defined partial edge incidence matrix of semigraph and author [2] defined the adjacency matrix of semigraph. In this paper strong circuit matrix and strong path matrix of semigraph are defined. The results of circuit matrix and path matrix of graph [4,6] are generalized in this paper.

2. Preliminaries

Definition 2.1. [5] A semigraph G is an ordered pair $(V; X)$ where V is a non-empty set, whose elements are called *vertices* of G and a set X is a set of *n - tuples*, called *edges* of G, of distinct vertices, for various $n \geq 2$, with the following conditions :

SG1: Any two edges have at most one element in common.

SG2: Two edges $(u_1; u_2 ; \ . \ . \ . \ u_n)$ and $(v_1; v_2; \ . \ . \ . \ ; \ v_m)$ are considered to be equal if and only if

i) $m = n$ and

ii) either $u_i = v_i$ or $u_i = v_{n-i+1}$ for i = 1, 2, 3, . . n

Thus the edge $(u_1; u_2; . . . . u_n)$ is the same as the edge $(u_n; u_{n-1}; . . . u_1)$.

Let $G = (V; X)$ be semigraph and $E = (v_1; v_2; . . . ; v_{n-1}; v_n)$ is an edge of G. Then the vertices v_1 and v_n are called the end vertices, represented by thick dots, the vertices v_2; . . . . ; v_{n-1} are called the middle vertices or m-vertices, represented by small hollow circles. A vertex v in G which appears as end vertex of one edge and middle vertex of the other edge is known as the middle-cum-end (m, e) vertex represented by a small tangent to the hollow circle of middle vertex.

Example 2.2. Let $G = (V; X)$ be a semigraph (Figure 1) , where $V = (v_0; v_1; v_2; v_3; v_4 ,v_5; v_6; v_7; v_8)$ and $X = ((v_0; v_1; v_2); (v_1; v_3; v_4); (v_4; v_5); (v_5; v_6; v_7); (v_2; v_7; v_8))$ In G, $v_0; v_2; v_4; v_5; v_8$ are end vertices, v_3 and v_6 are middle vertices, v_1 and v_7 are middle-cum-end vertices.

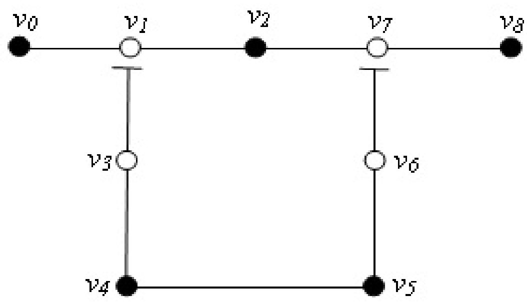

Figure 1: Semigraph G

Definition 2.3. [5] A *subedge* of an edge $E = (v_1; v_2; . . . v_n)$ is a *k-tuple* $E' =(v_{i1} ; v_{i2} ; . . . v_{ik})$ where $1 \leq i1 < i2 < . . < ik \leq n$ or $1 \leq ik < i(k+1) < :: < il \leq n$.

Definition 2.4. [5] A *partial edge* of $E = (v_1; v_2; . . . v_n)$ is a $(j - i + 1)$-tuple $E'(v_i; v_j) = (v_i; v_{i+1}; . . . v_j)$, where $1 \leq i \leq n$.

Definition 2.5. [5] *fs-edge* is an edge which is either a full edge or a subedge and *fp-edge* is an edge which is either a full edge or a partial edge.

Definition 2.6. [1] Let $E = (v_1; v_2; . . . v_n)$ be an edge of a semigraph G. Two subedges $S_j = (v_{j1}; v_{j2} ; . . . v_{jl})$ where $1 \leq j1 < j2 < . . . < jl \leq n$ and $S_k = (v_{k1}; v_{k2} ; . . . ; v_{km})$ where $1< k1 < . . . < km \leq n$ of E are said to be *consecutive subedges* if $v_{jl} = v_{k1}$.

Two partial edges $P_j = (v_j ; v_{j+1}; v_{j+2} ; . . . v_{j+l})$ and $P_k = (v_k ; v_{k+1}; . . . v_{k+m})$ of E are said to be consecutive partial edges if $v_{j+l} = v_{k+m}$

An edge $E = (v_1; v_2; . . . v_n)$ has n - 1 partial edges of cardinality two namely $P_1 = (v_1;v_2)$; $P_2 = (v_2 ; v_3)$; . . $P_{n-1} = (v_{n-1}; v_n)$ such that P_i and P_{i+1} are consecutive partial edges for i = 1; 2; . . . n - 2.

The partial edge $P_1 = (v_i ; v_{i+1})$ is e-partial edge if both v_i and v_{i+1} are end vertices and forms an edge. It is mm-partial edge if both v_i and v_{i+1} are middle vertices and me - partial edge if one vertex is middle and other is end.

Definition 2.7. [5] A *walk* in a semigraph G is an alternating sequence of vertices and fs-edges $v_0E_1v_2E_2 \; . \; . \; . \; v_{n-1}E_nv_n$ beginning and ending with vertices, such that v_{i-1} and v_i are the end vertices of the fs-edge E_i, $1 \leq i \leq n$.

A v_0 - v_n walk is a *trail* if any two fs-edges in it are disjoint. Note that in a trail vertices may be repeated.

A v_0 - v_n *path* is a v_0 - v_n trail in which all the vertices are distinct.

A *cycle* is a closed path.

A v_0 - v_n path is an *s-path* (or a strong path) if all its fs-edges are fp-edges. Otherwise, it is a *w-path* (or a weak path). Similarly, we define an s-cycle and a w-cycle.

In Figure 1, $v_0; v_2; v_1; v_4; v_5; v_6; v_7$ is w-path, $v_0; v_1; v_3; v_4; v_5; v_6; v_7$ is s-path, $v_0; v_2; v_1; v_4; v_5; v_6; v_7; v_2; v_0$ is w-cycle and $v_1; v_3; v_4; v_5; v_6; v_7; v_2; v_1$ is s-cycle.

Definition 2.8. [8,11,14] *Galois Field of prime power $GF(2^2)$* is the field of polynomials of degree less than *2* over $GF(2)$ modulo $(\alpha^2 + \alpha + 1)$ contains four elements 0; 1; α; $\alpha^2 = \alpha + 1$ where α is a root of the polynomial x^2+x+1 (with coefficients in $GF(2)$). The addition and multiplication operation on $GF(2^2)$ are as shown in the Table 1 and Table 2.

+	0	1	α	α^2
0	0	1	α	α^2
1	1	0	α^2	α
α	α	α^2	0	1
α^2	α^2	α	1	0

Table 1: Addition operation

×	0	1	α	α^2
0	0	0	0	0
1	0	1	α	α^2
α	0	α	α^2	1
α^2	0	α^2	1	α

Table 2: Multiplication Operation

Definition 2.9. [7] The *partial edge incidence matrix B* of a semigraph G is a matrix of order $n \times m$, where n is number of vertices and m is number of consecutive partial edges P_i of cardinality 2 of semigraph G, is defined as

b_{ij} = 1, if e – partial edge or me – partial edge P_j is incident on end vertex v_i

= α, if me – partial edge P_j is incident on middle vertex v_i

$= \alpha^2$, if mm – partial edge P_j is incident on middle vertex v_i

$= 0$, otherwise

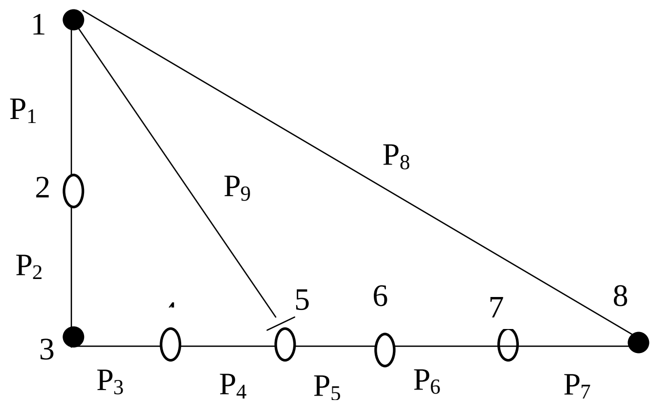

Figure2: Semigraph *G*

Example 2.10. For the semigraph *G* (Figure 2), the partial edge incidence matrix *B(G)* is

$$B(G)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & \alpha^2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \alpha^2 & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^2 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

3. Main results

Now we define the strong circuit matrix of semigraph.

Definition 3.1. The strong circuit matrix $C = [c_{ij}]$ of a semigraph *G* is a matrix of order $q \times m$, where q is number of strong circuit of semigraph *G* and m is number of consecutive partial edges P_i of cardinality 2 of semigraph *G*, is defined as

$c_{ij} = 1$, if i^{th} circuit include e-partial edge P_j ;

$= \alpha$, if i^{th} circuit include mm-partial edge P_j ;

$= \alpha^2$, if i^{th} circuit include me-partial edge P_j ;

$= 0$, otherwise

The above definition is illustrated in Example 3.2

Example 3.2. For the semigraph G (Figure 2), the strong circuits are

$C_1 = (P_1; P_2; P_3; P_4; P_9)$, $C_2 = (P_5; P_6; P_7; P_8; P_9)$ and $C_3 = (P_1; P_2; P_3; P_4; P_5; P_6; P_7; P_8; P_9)$

The corresponding strong circuit matrix is

$$C(G) = \begin{bmatrix} \alpha^2 & \alpha^2 & \alpha^2 & \alpha & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \alpha & \alpha & \alpha^2 & 1 & 1 \\ \alpha^2 & \alpha^2 & \alpha^2 & \alpha & \alpha & \alpha & \alpha^2 & 1 & 0 \end{bmatrix}$$

Remark 3.3. In case of Strong circuit matrix,

- The number of nonzero entries in each row is equal to the number of partial edges of cardinality 2 in the corresponding circuit.
- A column of all zero corresponds to a non circuit edge.
- The permutation of any two columns in a strong circuit matrix corresponds to relabeling of partial edges.
- The permutation of any two rows in a strong circuit matrix corresponds to relabeling of strong circuits.

The following theorem characterizes the strong circuit matrix of a semigraph.

Theorem 3.4. Let C and B be, respectively, the strong circuit matrix and the partial edge incidence matrix of semigraph (as per the definition 3.1 and the definition 2.9) whose columns are arranged using the same order of partial edges. Then the product BC^T or CB^T (with respect to $GF(2^2)$) is the matrix containing elements zero or α.

Proof: Consider a vertex v and a strong circuit C_i in the semigraph G. Then either v is in C_i or v is not in C_i . If v is not in C_i , there is no partial edge of cardinality two in C_i that is incident on v. On the other hand if v is in C_i, the number of those partial edges in the circuit C_i that are incident on v is exactly two.

Consider i^{th} row in B and j^{th} row in C. Since partial edges are arranged in the same order, the nonzero entries in the corresponding positions occur only if the particular partial edge is incident on the i^{th} vertex and is also in the j^{th} circuit.

If the i^{th} vertex is not in the j^{th} circuit, then the dot product of the two rows is zero. If the i^{th} vertex is in the j^{th} circuit, then the following ten cases arise.

Case (i) Let i^{th} vertex v is an end vertex and let two partial edges incident on v in circuit C_i are both e-partial edges then the corresponding element in the matrix BC^T is $(1.1) + (1.1) = 1 + 1 = 0$ with respect to $GF(2^2)$.

Case (ii) Let i^{th} vertex v is an end vertex and let two partial edges incident on v in circuit C_i are both me-partial edges then the corresponding element in the matrix BC^T is $(1.\alpha^2) + (1.\ \alpha^2) = \alpha^2 + \alpha^2 = 0$ with respect to $GF(2^2)$.

Case (iii) Let i^{th} vertex v is an end vertex and let the out of the two partial edges incident on v in circuit C_i one be, e-partial edge and other one be, me-partial edge then the corresponding element in the matrix BC^T is (1.1) + (1. α^2) = 1 + α^2 = α with respect to $GF(2^2)$.

Case (iv) Let i^{th} vertex v is the middle vertex and two partial edges incident on v in circuit C_i are both me-partial edges then the corresponding element in the matrix BC^T is (α. α^2) + (α.α^2) = α^3+ α^3= 1 + 1 = 0 with respect to $GF(2^2)$.

Case (v) Let i^{th} vertex v is the middle vertex and two partial edges incident on v in circuit C_i are me-partial edge and mm-partial edge (represented by α) then the corresponding element in the matrix BC^T is (α. α^2) + (α^2.α) = α^3+ α^3= 1 + 1 = 0 with respect to $GF(2^2)$.

Case (vi) Let i^{th} vertex v is the middle vertex and two partial edges incident on v in circuit C_i are both mm -partial edges then the corresponding element in the matrix BC^T is (α^2.α) + (α^2.α) = α^3+ α^3= 0 with respect to $GF(2^2)$.

Case (vii) Let i^{th} vertex v is the middle-cum-end vertex and two partial edges incident on v in circuit C_i are me-partial edge and e-partial edge then the corresponding element in the matrix BC^T is (α. α^2) + (1.1) = α^3+ 1 = 1 + 1 = 0 with respect to $GF(2^2)$.

Case (viii) Let i^{th} vertex v is the middle-cum-end vertex and two partial edges incident on v in circuit C_i are both me-partial edges then the corresponding element in the matrix BC^T is (α. α^2) + (1. α^2) = 1 + α^2 = α with respect to $GF(2^2)$.

Case (ix) Let i^{th} vertex v is the middle-cum-end vertex and two partial edges incident on v in circuit C_i are mm-partial edge and me-partial edge then the corresponding element in the matrix BC^T is (α^2.α) +(1. α^2) = α^3+ α^2 = 1+ α^2 = α with respect to $GF(2^2)$.

Case (x) Let i^{th} vertex v is the middle-cum-end vertex and two partial edges incident on v in circuit C_i are mm-partial edge and e-partial edge then the corresponding element in the matrix BC^T is (α^2.α) + (1.1) = α^3+ 1 = 1 + 1 = 0 with respect to $GF(2^2)$.

Therefore, in any case element in the matrix BC^T is 0 or α.

Similarly, it can be proved for CB^T.

Hence the theorem.

Example 3.5. For the semigraph G (Figure 2), let the partial edge incidence matrix $B(G)$ and strong circuit matrix $C(G)$ are as in Example 2.10 and Example 3.2 then it is clear that

$$BC^T = \begin{bmatrix} \alpha & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha & \alpha \end{bmatrix}$$

Corollary 3.6. Let C and B be, respectively, the strong circuit matrix and the partial edge incidence matrix of semigraph G whose columns are arranged using the same order of partial edges. If semigraph G is graph or semigraph with all edges having cardinality ≥ 3 and does not contain middle-cum-end vertex then the product BC^T or $CB^T = O$, null matrix (with respect to $GF(2^2)$).

Now we define the Strong Path Matrix of a Semigraph.

Definition 3.7. A strong path matrix is defined for a specific pair of vertices of a semigraph, say (x, y), and is denoted by $P(x, y)$. The rows in P corresponds to different paths between vertices x and y, and the columns correspond to the partial edges of G. Path matrix is defined as $P(x, y) =[p_{ij}]$, where

$$p_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ path include e-partial edge } P_j; \\ \alpha, & \text{if } i^{th} \text{ path include mm-partial edge } P_j; \\ \alpha^2, & \text{if } i^{th} \text{ path include me-partial edge } P_j; \\ 0, & \text{otherwise} \end{cases}$$

Example 3.8. Consider the all different strong paths between vertices 1 and 6 of semigraph G in Figure 2, $L_1 = (P_9; P_6)$, $L_2 = (P_8; P_7; P_6)$ and $L_3 = (P_1; P_2; P_3; P_4; P_5)$. The corresponding 3×9 path matrix is

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \alpha^2 & 1 & 0 \\ \alpha^2 & \alpha^2 & \alpha^2 & \alpha & \alpha & 0 & 0 & 0 & 0 \end{bmatrix}$$

Remark 3.9. In case of strong path matrix,

- A column of all zeros corresponds to an edge that does not lie on any path between x and y.
- A column of all nonzero entries corresponds to an edge that lies in every path between x and y.
- There is no row with all zeros.

The following theorem characterizes the strong path matrix of a semigraph.

Theorem 3.10. If the partial edges of a connected semigraph are arranged in the same order for columns of partial edge incidence matrix B and the strong path matrix $P(x, y)$, then the product $BP^T (x; y) = M$, (with respect to $GF(2^2)$) where the matrix M has two rows of x and y, are nonzero and the rest of n - 2 rows are zero.

Proof: Proof is similar to Theorem 3.4.

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