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Some Properties of Standard *n*-ideals of a Lattice

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Abstract. Standard and neutral elements (ideals) of a lattice were studied by many authors including Grätzer and Schmidt also see [1]. Generalizing the concept of standard ideals, Noor and Latif studied the standard n-ideals in [4,5]. In this paper the author have given some characterizations of these n-ideals and extended some of the results of [4,5]. They also includes a characterization of neutral n-ideals of a lattice when n is a neutral element.

Keywords: Standard element, Neutral element, Standard n-congruence

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1. Introduction

Standard and neutral elements (ideals) in a lattice L were studied by Grätzer and Schmidt in [2]. These concepts allow us to study a larger class of non-distributive lattices. Again in [4] and [5], Noor and Latif extended those concepts to study standard n-ideals in a lattice. In this paper we will examine some of the properties of standard n-ideals.

An element s of a lattice L is called a *standard element* if $x \wedge (s \vee y) = (x \wedge s) \vee (x \wedge y)$ for all $x, y \in L$. An element s is called neutral if

(i) it is standard in L, and

(ii) for all $x, y \in L$, $s \wedge (x \vee y) = (s \wedge x) \vee (s \wedge y)$.

For a fixed element n of a lattice L, a convex sublattice containing n is called an n-ideal. The idea of n-ideals is a kind of generalization of both ideals and filters of lattices. The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set-inclusion. Moreover, $\{n\}$ and L are respectively the smallest and the largest elements of $I_n(L)$. For any two n-ideals I and J of L it is easy to check that $I \wedge J = I \cap J = \{x \in L : x = m(i, n, j)$ for some $i \in I,\ j \in J\}$, where

$$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \text{ and}$$

$I \vee J = \{x \in L : i_1 \wedge j_1 \leq x \leq i_2 \vee j_2$, for some $i_1, i_2 \in I$ and $j_1, j_2 \in J\}$. The n-ideal generated by a finite number of elements $a_1, a_2, ..., a_m$ is called a *finitely generated n-ideal* denoted by $< a_1, a_2, ..., a_m >_n$, which is the interval

$$[a_1 \wedge a_2 \wedge ... \wedge a_m \wedge n, \ a_1 \vee a_2 \vee ... \vee a_m \vee n >_n].$$

The n-ideal generated by a single element a is called a principal n-ideal, denoted by $< a >_n = [a \wedge n, \ a \vee n]$. For detailed literature on n-ideals we refer the reader to consult [3].

An n-ideal of a lattice L is called a *standard (neutral) n-ideal* of L if it is a standard (neutral) element of $I_n(L)$. The following characterization of standard n-ideals is due to [4].

Theorem 1.1. Let n be a neutral element of a lattice L. An n-ideal S is a standard n ideal if and only if for any n-ideal K,

$$S \vee K = \{x \in L : x = (x \wedge s_1) \vee (x \wedge k_1) \vee (x \wedge n)\}$$
$$= \{x \in L : x = (x \vee s_2) \wedge (x \vee k_2) \wedge (x \vee n)\}$$ for some $s_1, s_2 \in S$ and $k_1, k_2 \in K$. □

We start this paper with the following characterization of standard n-ideals.

2. Main results

Theorem 2.2. Let n be a neutral element of a lattice L, An n-ideal S of a lattice L is standard if and only if $< a >_n \cap (S \vee < b >_n) = (< a >_n \cap S) \vee (< a >_n \cap < b >_n)$ for all $a, b \in L$.

Proof: Suppose S is standard. Then obviously the above relation holds.
Conversely, suppose above relation holds for all $a, b \in L$. Let K be an n-ideal of L and $x \in S \vee K$. Then $s_1 \wedge k_1 \leq x \leq s_2 \vee k_2$ for some $s_1, s_2 \in S$ and $k_1, k_2 \in K$. Now $n \leq x \vee n \leq s_2 \vee k_2 \vee n$ implies that
$x \vee n \in < x >_n \cap (S \vee < k_2 \vee n >_n) = (< x >_n \cap S) \vee (< x >_n \cap < k_2 \vee n >_n)$. Thus $x \vee n \leq t \vee r$ for some $t \in < x >_n \cap S$ and $r \in < x >_n \cap < k_2 \vee n >_n$. Then $t = (x \wedge s) \vee (x \wedge n) \vee (s \wedge n)$ for some $s \in S$ and $r \leq (x \vee n) \wedge (k_2 \vee n) = (x \wedge k_2) \vee n$, as n is neutral. Hence $x \vee n \leq (x \wedge s) \vee (x \wedge k_2) \vee n$, and so $x = x \wedge (x \vee n) \leq x \wedge ((x \wedge s) \vee (x \wedge k_2) \vee n)$ $= (x \wedge s) \vee (x \wedge k_2) \vee (x \wedge n) \leq x$. Thus $x = (x \wedge s) \vee (x \wedge k_2) \vee (x \wedge n)$.
By a dual proof of above we can prove that $x = (x \vee s') \wedge (x \vee k_1) \wedge (x \vee n)$ for some $s' \in S$. Therefore by Theorem 1.1, S is standard. □

In [5], Noor and Latif have proved that for a neutral element n of a lattice L, $< a >_n$ is standard if and only if $a \wedge n$ is dual standard and $a \vee n$ is standard. We extend the result for a finitely generated n-ideal.

Theorem 2.3. For $a_1, a_2,\dots, a_m, n \in L$, $<a_1, a_2,..., a_m>_n$ is standard if $a_1 \wedge n, a_2 \wedge n,..., a_m \wedge n$ are dual standard and $a_1 \vee n, a_2 \vee n,..., a_m \vee n$ are standard.

Proof: Let $I, J \in I_n(L)$. Suppose $x \in I \cap (<a_1, a_2,..., a_m>_n \vee J)$. Then $x \in I$ and $x \in <a_1, a_2,..., a_m>_n \vee J$. Then $a_1 \wedge ... \wedge a_m \wedge n \wedge j \le x \le a_1 \vee ... \vee a_m \vee n \vee j_1$ for some j, $j_1 \in J$. Thus, $x \vee n \le a_1 \vee ... \vee a_m \vee n \vee j_1$ which implies $x \vee n = (x \vee n) \wedge (a_1 \vee ... \vee a_m \vee n \vee j_1)$. Then using the standard of $a_1 \vee n,..., a_m \vee n,$ we have $x \vee n = ((x \vee n) \wedge (a_1 \vee n)) \vee ... \vee ((x \vee n) \wedge (a_m \vee n)) \vee ((x \vee n) \wedge (j_1 \vee n))$.

But $(x \vee n) \wedge (a_i \vee n) = m(x \vee n, n, a_i \vee n) \in I \cap <a_i \vee n>_n \subseteq I \cap <a_1,..., a_m>_n$.

Similarly, $(x \vee n) \wedge (j_1 \vee n) \in I \cap J$. Therefore, $x \vee n \in (I \cap <a_1,..., a_m>_n) \vee (I \cap J)$. Dually, using the dual standard property of $a_1 \wedge n,..., a_m \wedge n$ we can show that $x \wedge n \in (I \cap <a_1,..., a_m>_n) \vee (I \cap J)$, and so by convexity of n-ideal, $x \in (I \cap <a_1,..., a_m>_n) \vee (I \cap J)$.

Therefore, $I \cap (<a_1,..., a_m>_n \vee J) \subseteq (I \cap <a_1,..., a_m>_n) \vee (I \cap J)$. Since the reverse inclusion is trivial, so $I \cap (<a_1,..., a_m>_n \vee J) = (I \cap <a_1,..., a_m>_n) \vee (I \cap J)$, and hence $<a_1, a_2,..., a_m>_n$ is standard. ☐

By [1] we know that an element $n \in L$ is neutral if and only if for all $a, b \in L$, $(a \wedge b) \vee (a \wedge n) \vee (b \wedge n) = (a \vee b) \wedge (a \vee n) \wedge (b \vee n)$. Since this relation is self dual, so the dual condition of neutrality also implies the neutrality. So we have following extension of above theorem.

Theorem 2.4. For $a_1, a_2,..., a_m, n \in L$, $<a_1, a_2,..., a_m>_n$ is neutral if $a_1 \wedge n, a_2 \wedge n,..., a_m \wedge n$ and $a_1 \vee n, a_2 \vee n,..., a_m \vee n$ are all neutral elements in L.

Proof: Suppose $a_1 \wedge n, a_2 \wedge n,..., a_m \wedge n$ and $a_1 \vee n, a_2 \vee n,..., a_m \vee n$ are neutral. Then $a_1 \wedge a_2 \wedge ... \wedge a_m \wedge n$ and $a_1 \vee a_2 \vee ... \vee a_m \vee n$ are also neutral. By Theorem 1.4, $<a_1, a_2,..., a_m>_n$ is standard. So we need to show only the dual distributive property. Let $I, J \in I_n(L)$ and $x \in <a_1, a_2,..., a_m>_n \cap (I \vee J)$. Then $x \in <a_1, a_2,..., a_m>_n$ and $i_1 \wedge j_1 \le x \le i_2 \vee j_2$ for some $i_1, i_2 \in I$, $j_1, j_2 \in J$. So

$$
\begin{aligned}
x \vee n &\le (a_1 \vee a_2 \vee ... \vee a_m \vee n) \wedge [(i_2 \vee n) \vee (j_2 \vee n)] \\
&= [(a_1 \vee a_2 \vee ... \vee a_m \vee n) \wedge (i_2 \vee n)] \vee [(a_1 \vee a_2 \vee ... \vee a_m \vee n) \wedge (j_2 \vee n)] \\
&\in (<a_1, a_2,..., a_m>_n \cap I) \vee (<a_1, a_2,..., a_m>_n \cap J).
\end{aligned}
$$

A dual proof shows that $x \wedge n \in (<a_1, a_2,..., a_m>_n \cap I) \vee (<a_1, a_2,..., a_m>_n \cap J)$.

Hence, by convexity $x \in (< a_1, a_2, ..., a_m >_n \cap I) \vee (< a_1, a_2, ..., a_m >_n \cap J)$. Thus
$< a_1, a_2, ..., a_m >_m \cap (I \vee J) \subseteq (< a_1, a_2, ..., a_m >_n \cap I) \vee (< a_1, a_2, ..., a_m >_n \cap J)$
Since the reverse inclusion is trivial, so
$< a_1, a_2, ..., a_m >_n \cap (I \vee J) = (< a_1, a_2, ..., a_m >_n \cap I) \vee (< a_1, a_2, ..., a_m >_n \cap J)$
Therefore, $< a_1, a_2, ..., a_m >_n$ is dual standard and so it is neutral. □

By [4] we know that an n-ideal S of a lattice L is standard if and only if the relation $\Theta(S)$ defined by $x \equiv y\Theta(S)$ if and only if $x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$ and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$ for some $s, t \in S$ is the smallest congruence containing S as a class. We also know by [5] that for two standard n-ideals S and T, both $S \cap T$ and $S \vee T$ are standard. Moreover,

$$\Theta(S \cap T) = \Theta(S) \cap \Theta(T) \text{ and } \Theta(S \vee T) = \Theta(S) \vee \Theta(T).$$

The congruences of the form $\Theta(S)$ where S is a standard n-ideal, are known as standard n-congruences. Above relations show that the standard n-congruence's form a distributive lattice. We conclude the paper with the following result which is a generalization of [1, Example-15, page-150].

Theorem 2.5. For a neutral element n of a lattice L, the lattice of all standard n-ideals is isomorphic to the lattice of all standard n-congruences.

Proof: Between these two lattices consider the map $S \to \Theta(S)$. By above relations clearly this is a homomorphism and onto. So we need only to show that this is one-one. Suppose $\Theta(S) = \Theta(T)$ for two standard n-ideals S and T. Let $s \in S$. Then for any $t \in T$, $m(s,n,t) \in S$ Then $S \equiv m(s,n,t)\Theta(S) = \Theta(T)$. Since n is neutral, so $m(s,n,t) = (s \wedge t) \vee (s \wedge n) \vee (t \wedge n) = (s \vee t) \wedge (s \vee n) \wedge (t \vee n)$.
Thus, $s \wedge m(s,n,t) = s \wedge (t \vee n)$ and $s \vee m(s,n,t) = s \vee (t \wedge n)$..
Since $s \equiv m(s,n,t)\Theta(T)$, so
$s \wedge m(s,n,t) = ((s \wedge m(s,n,t)) \vee a) \wedge (s \vee m(s,n,t))$, and
$s \vee m(s,n,t) = ((s \vee m(s,n,t)) \wedge b) \vee (s \wedge m(s,n,t))$ for some $a, b \in T$ Thus,
$s \wedge (t \vee n) = ((s \wedge (t \vee n)) \vee a) \wedge (s \vee (t \wedge n))$ and
$s \vee (t \wedge n) = ((s \vee (t \wedge n)) \wedge b) \vee (s \wedge (t \vee n))$. Hence, $a \wedge t \wedge n \le s \wedge (t \vee n) \le t \vee n$ which implies $s \wedge (t \vee n) \in T$, by convexity of T. Similarly,
$t \wedge n \le s \vee (t \wedge n) \le b \vee t \vee n$ implies $s \vee (t \wedge n) \in T$. Since
$s \wedge (t \vee n) \le s \le s \vee (t \wedge n)$, so by applying the convexity again $s \in T$. This implies $S \subseteq T$. Similarly $T \subseteq S$ and so $S = T$. Therefore, above mapping is one-one and hence it is an isomorphism. □

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