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On Family of Ehrhart Polynomials and Counterexamples of the Conjecture of Beck and Al.

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Abstract. The aims of goal of this paper is to study a family of Ehrhart polynomials for find a special polynomial verifying the conjecture of Beck and al. Then we provide some contribution on counterexamples for this conjecture.

Keywords: Family of Ehrhart polynomials, Special polynomial, Semigeometric sequence.

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1. Introduction

For several decades, it was discovered that the whole polytopes theory had very close links with the geometry of algebraic varieties. In fact, this will plunge the polytopes theory to problems in the wider world of the geometry of projective varieties. Ehrhart work on Diophantine equations that gave birth to the now Ehrhart polynomials on convex polytopes. The polyhedron associated with a linear Diophantine system is used for solve the combinatorial problem cf. [4] and [5]. Others use the convex polytopes in search of solving methods mathematical problems. At the latest, many researchers have worked on roots of Ehrhart polynomials and many of them have found the counterexamples of Beck and al. conjecture, see [2] " All roots α_i of Ehrhart polynomials of an integral convex polytopes of dimension d satisfy the relation $-d \leq Re(\alpha_i) \leq d-1$ for all i, with $\alpha_i \in \mathbb{C}$ where $Re(\alpha_i)$ name the real part of α_i ". In 2006, [7] analysed the behavior of the roots of general polynomials satisfying the conditions of Stanley Theorem and compared the behavior of known roots of all of the Ehrhart polynomials. In the same way, they gave a possible counterexample of the previous conjecture. In 2011, [8] had worked especially on the exhaustive calculation of Ehrhart polynomials. This led them, among other things, to presente two new corrective conjectures of the conjecture of Beck and al. on the Ehrhart polynomials roots of convex polytopes integrals of dimension d by a numerical method using the Maple and Maxima softwares. But in 2012, [10] have demonstrated that if the cycle length is 127 then Ehrhart polynomials have a root whose real part is superior or equal to the dimension with the help a smooth polytope of Fano, in other words, they were able to show that is a counterexample to both conjectures of [8]. In this paper, we propose to study a family of Ehrhart polynomials using an algebraic method. By direct calculation using Maple 13 software in "work sheet" mode, we find some properties of these polynomials. We show that the dominant coefficients of the family of Ehrhart polynomials are equal to $m+1$ and the gcd of all coefficients other than the dominant term of this family is a divisor of $m+1$. We get that the constant term of any family of Ehrhart polynomials dimension d ($d \geq 3$) is decomposable into decreasing factor products. Any family of Ehrhart polynomials $g_{m,d,k}$ has a unique special polynomial having the same dimension and degree than this family of polynomials. This special polynomial verifies the conjecture Beck and al. for a minimum dimension d ($d \geq 17$) (resp. d ($d \geq 20$)) for d is odd (resp. d is pair). We give some details of Higashitani counterexamples to the conjecture of Beck and al., that is, considering a convergent special sequence semigeometric of the family of Ehrhart polynomials of dimension d and degree k, there exists always a threshold interval of integer m containing counterexamples. Examples are given to illustrate our results. l.

2. Study of Ehrhart polynomials

2.1. Preliminary and notations

We denote by f_P cf.[3], if there exists not notation confusion, the Ehrhart polynomials such that $f_P(n) = card(nP \cap \mathbb{Z}^d)$, for all $n \geq 1$. We study a family of Ehrhart polynomials of parameters m, d, k defined by

$$g_{m,d,k}(n) = \prod_{j=d-k+1}^{d} (n+j) + m\prod_{j=0}^{k-1}(n-j). \tag{1}$$

Let's study a family of Ehrhart polynomials of parameters m, d, k defined by

$$g_{m,d,k}(n) = \prod_{j=d-k+1}^{d} (n+j) + m\prod_{j=0}^{k-1}(n-j). \tag{2}$$

We will denote by p_S the derived special polynomial of the family of Ehrhart polynomials. In the Theorem 2.1 of paragraph 2 of [6], Higashitani constructs following the clever manner a convex polytope with integer vertices of dimension d, this Ehrhart polynomials verify:

$$(1-\lambda)^{d+1}\sum_{n\geq 0} f_P(n)\lambda^n = 1 + m\lambda^k, \tag{3}$$

where m, d, k are integers verifying $m \geq 1,\ d \geq 2,\ 1 \geq k \geq \left[\frac{d+1}{2}\right]$. We deduce that

$$\sum_{n\geq 0} f_P(n)\lambda^n = (1-\lambda)^{-d-1}(1+m\lambda^k) = \left[\sum_{n\geq 0}\binom{d+n}{d}\lambda^n\right](1+m\lambda^k) \tag{4}$$

Hence we have

$$f_P(n) = \frac{(n+d)\ldots(n+1)}{d!} + m\frac{(n+d-k)\ldots(n+1-k)}{d!} \tag{5}$$

When k verifies $1 \geq k \geq \left[\frac{d+1}{2}\right]$, we can write

$$f_P(n) = \frac{1}{d!} g_{m,d,k}(n) \prod_{j=1}^{d-k} (n+j) \qquad (6)$$

$$where \quad g_{m,d,k}(n) = \prod_{j=d-k+1}^{d} (n+j) + m \prod_{j=0}^{k-1} (n-j) \qquad (7)$$

is a family of polynomials in the variable n of degree k, with k positive integer, to integers coefficient which roots are considered in paragraph 3 of [6]. This family of polynomials belongs to the set of polynomial fonctions in $\mathrm{R}[x]$, so all principle results of polynomials theory in $\mathrm{R}[x]$, that is, euclidian division, Bezout theorem, roots of polynomial, factorisation, d'Alembert theorem, etc... apply also to the family of Ehrhart polynomials. To make the study easier, we distinguish two types of the family of polynomials according to the parity of dimension d : $g_{m,d,k}$ for the family of Ehrhart polynomials of odd dimension and $g_{m,d,k'}$ for the pair dimension.

Definition 2.1. We call family of Ehrhart polynomials the polynomials in n variable, of dimension d and degree k (resp. k') if d is odd (resp. pair), defined by:

$$For\ d\ is\ odd: g_{m,d,k}(n) = \prod_{j=d-k+1}^{d} (n+j) + m \prod_{j=0}^{k-1} (n-j) \qquad (8)$$

$$= \sum_{i=0}^{k} a_i n^i, where\ \ m \geq 1, d \geq 2\ \ and\ k = \frac{d+1}{2} \qquad (9)$$

$$\text{For d is pair} \quad g_{m,d,k'}(n) = \prod_{j=d-k'+1}^{d} (n+j) + m \prod_{j=0}^{k'-1} (n-j) \qquad (10)$$

$$= \sum_{i=0}^{k'} a_i n^i, where\ \ m \geq 1, d \geq 2\ \ and\ k' = \frac{d}{2}. \qquad (11)$$

The dominant coefficient of this polynomial is the one by monome of highest degree as well as the coefficient of monome of zero degree is called by its constant term noted by J_d^k (resp. $J_d^{k'}$) or a_0. To put a prominent position the counterexamples of the conjecture of Beck and al, Higashitani considere more particularly the limit cases where $k = \frac{d+1}{2}, k' = \frac{d}{2}$. And in order to treat various examples, we use the following Maple code:

```
> restart;
> with(plots):
> Digits:=16:
```

```
> ehrhart:=proc(m,d,k) mul(n+i,i=1,…..,.d)+m*mul(n-k+i,i=1,…,..d); end proc:
> fung:=proc(m,d,k) mul(n+j,j=d-k+1,…, d)+m*mul(n-j,j=0,…,..k-1); end proc:
> g:=ehrhart(m,d,k)
> Iz:=[fsolve(fung(m,d,k),n,complex)];
```

Properties 2.2.
(a) The dominant coefficient of the family of Ehrhart polynomials $g_{m,d,k}$ is $m+1$.
(b) The gcd of all coefficients except that of the coefficient of the dominant term of $g_{m,d,k}$ is a divisor of $m+1$.
(c) The constant term of the family of Ehrhart polynomials is given by $J_d = \prod_{j=d-k+1}^{d} j$ (resp. $J_d = \prod_{j=d-k'+1}^{d'} j$) with $k = \left[\frac{d+1}{2}\right]$ if d is odd (resp. $k' = \left[\frac{d}{2}\right]$ if d is pair).

Proof.
(a) Result from the expression of $g_{m,d,k}$.
(b) Let $g_{m,d,h}$ be a family of Ehrhart polynomials of n variable, h degree and d dimension in $\mathbb{R}^d$. This family can be written to $g_{m,d,h}(n) = a_h n^h + a_{h-1} n^{h-1} + \ldots + J_d$ with $a_h = m+1$ and $a_0 = J_d$. Suppose that $gcd\left(a_{h-1}, \ldots, J_d\right) = l$. If $l = 1$ then we have a trivial case, hence the family $g_{m,d,h}$ is a primitive polynomial. If $l \neq 1$ then:

- for $a_h = l$ we have $gcd\left(a_h, l\right) = l$ so l divides $m+1$,
- for $a_h > 1$, a_h is decomposable in product of first factors such as

$a_h = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, with n integer, $\alpha_i \in \mathbb{N}$, p_i first numbers; $0 \leq i \leq n$ and when $l = p_i$ where $0 \leq i \leq n$ we show that l divide $m+1$ whereas for $l \neq p_i$, it comes to the trivial case, that is $pgcd(a_h, l) = 1$.
(c) Taking $n = 0$ from (4), we have the result.

Remark 2.3. The sum of constant terms of the family of Ehrhart polynomials of dimension d ($d \geq 3$) is different from the constant term of the sum of dimensions of this family, that is, $\sum_{d=3}^{n} J_d \neq J_{\sum_{d=3}^{n} d}$ where $n > d$.

Example 2.4.

$m=1, d\geq 3$	3	4	5	6	7	$\ldots$	12
J_d	6	12	60	120	840	$\ldots$	665280
$\sum_{d=3}^{5} J_d$	6	18	78	198	1038	$\ldots$	$\ldots$
$J_{\sum_{d=3}^{5} d}$	6	840	665280	$\ldots$	$\ldots$	$\ldots$	$\ldots$

Proposition 2.5. The constant term J_d of all family of Ehrhart polynomials of dimension d ($d \geq 3$) is decomposable in decreasing factor products.

Proof. This results from (6) and (7) expressions for $n = 0$. Because,

$J_d = \prod_{j=d-k+1}^{d} j = \frac{d!}{(d-k)!}$ where k verifies the condition on the parity of the dimension d.

2.2. On the conjecture of Beck and al.

We remind that the conjecture of Beck and al, cf.[6] given by the following relation

$$-d \leq Re(\alpha_i) \leq d-1 \qquad (12)$$

, for all nonnegative integer i, where $\alpha_i \in \mathbb{C}$ and, $Re(\alpha_i)$ refers to their real part which are the roots of Ehrhart polynomials of degree d. The counterexamples on this conjecture depend on the minimal dimension d of these polynomials and of the threshold interval $[a_d, b_d]$ where m take its values, with a_d is the lower bound and b_d the upper bound of this interval.

Results of Higashitani 2.6

• For $9 \leq m \leq 173$, the minimum dimension for which the Higashitani method give a counterexample is $d = 15$ but for $174 \leq m \leq 200$, we obtain $d = 17$. So there is no hope of finding a value of $d \leq 14$ for a counterexample by increasing beyond 200 the value of m.

• For $m = 9$, Higashitani sais to have proven that $g_{9,m,k}$ admits a root of superior real part to $d-1$ for $15 \leq d \leq 100$, and same for $17 \leq d \leq 100$ this maximum real part is superior to d. It turns out that when d increases beyond 100, the real part is strongly increasing. Higashitani was capable of going beyond the dimension $d = 100$ with his numerical programs.

We find the same results that as Higashitani's results. Actually, we find some others result, for example: $174 \leq m \leq 621$ pour $d = 17$ et $d = 18$, we have also the counterexamples if $m \in [6{,}515]$, etc... We can make the programs using the Maple 13 code "work sheet" mode below and we execute after that :

Example 2.7.

• For the family of Ehrhart polynomials $g_{174,17,9}$:

```
> restart;
> with(plots):
> Digits:=16:
> ehrhart:=proc(m,d,k) mul(n+i,i=1....d) +m*mul(n-k+i,i=1.....d); end proc:
fung:=proc(m,d,k) mul(n+j,j=d-k+1...d) + m*mul(n-j,j=0.....k-1); end proc:
> g:=ehrhart(174,17,9)
> Iz:[fsolve(fung(174,17,9),n,complex)];
```

• The same for $g_{621,17,9}$ after replacing m by 621 and keeping the compatible values of d and k. And so on.

We can also apply the same programs to the case of pair dimension, for example, $g_{6,18,9}$ and $g_{422,18,9}$ so that $g_{423,18,9}$. Consequently, there are polynomials who verify the conjecture and others which do not verify, so therefore, we try to characterize the corresponding minimal dimension and threshold interval by which we obtain the counterexamples to the conjecture of Beck and al. An algebraic method with decimal constant, allows to calculate these dimensions and threshold intervals.

2.3. Semigeometric and special semigeometric sequences

Definitions 2.8. Let $(v_i)_{1\le i\le n}$ be a numerical sequence.

1. The sequence $(v_i)_{1\le i\le n}$ is called semigeometric sequence of 1 order if it's in geometric progression of reason q, with $q=\dfrac{v_{i+1}}{v_i}$.

2. The sequence $(v_i)_{1\le i\le n}$ is said semigeometric of 2 orders if there are exactly 2 and 2 different reasons q_1 et q_2 such that $q_1=\dfrac{v_{i+1}}{v_i}$ and $q_2=\dfrac{v_{j+1}}{v_j}$, for all $i\neq j$.

3. We call semigeometric sequence of k orders with $k>1$ a numerical sequence $(v_i)_{1\le i\le n}$ if there are $k\le n$ different reasons 2 to 2 such that for all $k_1,k_2\le n, i\neq j;\quad k_1\neq k_2$ we have $q_{k_1}=\dfrac{v_{i+1}}{v_i}$ and $q_{k_2}=\dfrac{v_{j+1}}{v_j}$.

Proposition 2.9. Let $(v_i)_{1\le i\le n}$ be a semigeometric sequence of k order, with k nonzero positive integer. The following assertions are equivalent:

- i) Each term v_i of the sequence $(v_i)_{1\le i\le n}$ is nonzero,
- ii) The reasons q_k of $(v_i)_{1\le i\le n}$ are nonzero reals.

Proof. $i)\Rightarrow ii)$. Suppose that each term v_i of the sequence $(v_i)_{1\le i\neq n}$ is a nonzero real, then in accordance with the Definition 2.8., for all $k_1,k_2\le n$, $i\neq j, k_1\neq k_2, q_{k_1}=\dfrac{v_{i+1}}{v_i}$

and $q_{k_2} = \frac{\nu_{j+1}}{\nu_j}$ that is to say $q_k = \frac{\nu_{i+1}}{\nu_i} = \frac{\nu_{i+1}}{1} \times \frac{1}{\nu_i}$ (for all $k \le n$), with ν_i and ν_{i+1} are nonzero reals. Then $\nu_{i+1} \times \frac{1}{\nu_i}$ is a nonzero real because $(\mathsf{R},+,\times)$ is a corps. That proves that q_k is a non nul real.

$ii) \Rightarrow i)$. We suppose that $q_k = \frac{\nu_{i+1}}{\nu_i}$ is a non null real, as $(\mathsf{R},+,\times)$ is a corps then $q_k = \nu_{i+1} \times \frac{1}{\nu_i}$ is also a nonzero real. Therefore each term ν_i of the sequence $(\nu_i)_{1 \le i \le n}$ is a nonzero real. Hence the proof.

Proposition and definition 2.10. Suppose that the semigeometric sequence $(\nu_i)_{1 \le i \le n}$ is k order with $k \ge 3$. There exists a sequence $(q_k)_{1 \le k \le n}$ called "special semigeometric of r order with $r < k$" verifying the assertions of the Proposition 2.9.

Proof. Let $(\nu_i)_{1 \le i \le n}$ be a semigeometric sequence of k order ($k \ge 3$). By Definition 2.8. there are k ($k < n$) different reasons 2 to 2 which enables us to obtain a sequence $(q_k)_{1 \le k \le n}$ r order ($r < k$) of general term $q_k = \frac{\nu_{i+1}}{\nu_i}$. We distinguish the cases:

• If $k = 3$, (ν_i) is a semigeometric sequence of 3 order such that there are different three reasons 2 to 2 q_k with $1 \le k \le 3$. Considering the Definition 2.8. those three reasons form a special semigeometric $(q_k)_{1 \le k \le 3}$ of 2 order.

• If $k \ge 3$, (ν_i) is a semigeometric sequence of k order ($k \ge 3$) verifying the k different reasons of 2 to 2 constitute a special semigeometric $(q_k)_{1 \le k < n}$ of r ordre with $r < k$ by Definition 2.8. and in reasonning by recurrence. That finishes the proof.

Theorem 2.11. All special semigeometric sequence is semigeometric.

Proof. Let $(q_i)_{1 \le i < k}$ be a special semigeometric sequence of r order where $r < k$ verifying $\alpha_r = \frac{q_{r+1}}{q_r}$, it gives that the α_r with $1 \le r < k$ are possible different reasons of the special semigeometric. For $r = 1$, there exists a nonzero real $\alpha_1 = \frac{q_2}{q_1}$ such that the terms q_1 and q_2 form a semigeometric of 1 order. For any r with ($2 \le r < k$), by Proposition 2.9 we have $\alpha_r = \frac{q_{r+1}}{q_r} \ne 0$. If there are two different nonzero and nonnegative integers r_1 et r_2 strictly inferior to k such that $\alpha_{r_1} = \dfrac{q_{r_1+1}}{q_{r_1}}$ and $\alpha_{r_2} = \dfrac{q_{r_2+1}}{q_{r_2}}$ then we obtain a semigeometric sequence $(q_i)_{1\le i<k}$ of 2 order. Suppose that now $r \in \{3,\dots,k-1\}$, a reasonning by recurrence drives us to find a semigeometric sequence $(q_i)_{1\le i<k}$ of r order where ($r<k$). There is the proof.

In the next, suppose that all sequence $(q_i)_i$ is a special semigeometric of r order, where r is nonzero and nonnegative integer.

Corollary 2.12. From all sequence $(q_i)_i$ we can construct another special semigeometric $(\alpha_s)_{1\le s<r}$ of s order such that $\alpha_s = \dfrac{q_{s+1}}{q_s}$.

Proof. Let $(q_i)_{1\le i<k}$ be a special semigeometric of r order, (r is nonzero and nonegative integer). By Theorem 2.11, $(q_i)_{1\le i<k}$ is a semigeometric sequence of r order then in accordance with the Proposition 2.10, there exists a special semigeometric $(\alpha_i)_{1\le i<r}$ of s order verifying the assertions of the Proposition 2.9 such that $\alpha_s = \dfrac{q_{s+1}}{q_s}$. It finishes the proof.

Proposition 2.13. If the sequence $(\alpha_s)_{1\le s<r}$ constructed from $(q_i)_i$ verifying the condition of Corollary 2.12 is a finite sequence of terms then from a certain rank, $(q_i)_i$ is stationary.

Proof. Let $(\alpha_i)_{1\le i<r}$ be a sequence constructed from $(q_i)_{1\le i\le n}$ with $r<n$. $(\alpha_i)_{1\le i<r}$ is a special semigeometric sequence satisfying the condition of Corollary 2.12. Then the sequence $(q_i)_{1\le i\le n}$ is also a special semigeometric of s order. But $(\alpha_i)_{1\le i<r}$ is a finite sequence of terms (by hypothesys) thus $(q_i)_{1\le i\le n}$ as well. It is clear that if the sequence $(q_i)_i$ is stationary from a certain rank then it converges. Suppose that $(q_i)_i$ converges to a real limit l. Let be $\varepsilon > 0$, there exists a nonnegative integer N in such a way that for all nonnegative integer i with $i \ge N$ we have $|q_i - l| < \dfrac{\varepsilon}{2}$. Let be a nonnegative integer i where $i \ge N$, we have $|q_i - q_N| = |(q_i - l) + (l - q_N)| \le |q_i - l| + |q_N - l| < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$. Since q_i and q_N are reals, as a result $q_i = q_N$. That proves that the sequence $(q_i)_{1\le i\le n}$ is stationary.

Remark 2.14. The special semigeometric sequence $(\alpha_s)_s$ constructed from the special semigeometric sequence $(q_i)_i$ maybe non-monotonous. For example in the case where the dimension d is odd.

Obtaining counterexamples of the conjecture (12) depends only on the minimal dimension d and threshold interval of m of the polynomials. Precisely to suitable dimension d with $d \geq 15$ (resp. $d \geq 18$) for odd dimension (resp. pair), we have counterexamples for m belongs to the threshold interval though polynomials verify the conjecture (12) with the same dimension and outside this threshold interval of m. An algebraic method with only one decimal constant enables to calculate the minimum dimensions and the threshold interval of m for which we obtain the counterexamples.

Proposition 2.15. Let d be a suitable dimension verifying the above condition of $g_{m,d,k}(n)$ (resp $g_{m,d,k'}(n)$), m nonzero and nonnegative integer and, $[a_d, b_d]$ a threshold interval, where a_d and b_d nonzero and nonnegative integers. If m is not belong to $[a_d, b_d]$ then the polynomials $g_{m,d,k}(n)$ (resp. $g_{m,d,k'}(n)$) verify the formula (12) in [2]. If m belong to $[a_d, b_d]$ we find that the conjecture is not verified.

Proof. Suppose that the nonzero and nonnegative integer m belongs to threshold interval $[a_d, b_d]$ where a_d, b_d nonzero and nonnegative integers with $a_d \geq 9$, d a suitable dimension or minimum dimension verifying: if d is odd then $d \geq 15$ and if d is pair we have $d \geq 18$. A worksheet on Maple 13 proves that all roots of the family of Ehrhart polynomials do not verify the conjecture of Beck and al. (12) cf. [2]; that is, we are in counterexamples cases. Otherwise, that is the integer m does not belong to the threshold interval $[a_d, b_d]$ where a_d, b_d are nonzero and nonnegative integers, the formula (12) is verified. There is the proof.

We give some examples to illustrate this Proposition.

Example 2.16. If $d = 15$ and $m \geq 174$, the conjecture of Beck and al. is verified.
If $d = 17$ and $m \in [9,173]$ with $m > 621$ then no counterexamples appear.
If $d = 19$ with $m \in [622, 1882]$, so there are counterexamples.
For $d = 49$ and $m > 1971766754$, there is not counterexamples.
For $d = 51$ and $m \in [1971766755, 4766545270]$ so we find the counterexamples to the conjecture.

We give some examples of the family of Ehrhart polynomials that for each value of the pair dimension $d \geq 18$, we obtain a threshold interval in which we find the counterexamples to the conjecture of Beck and al.

Example 2.17. For $d = 18$, on a $[6,515]$ the threshold interval who contains m.

For $d = 20$, the corresponding threshold interval is $[516,1590]$.

If $d = 22$, we have $[1591,4516]$ the threshold interval of the integer m verifying of counterexamples.

When $d = 24$, $m \in [4517,12262]$ then there is always the counterexamples to the conjecture.

If $d = 26$, we obtain $[12263,32344]$ the threshold interval who contain m does not verifying the conjecture.

When $d = 28$, $m \notin [4517,12262]$ then we have a verified conjecture.

For $d = 30$, inside to $[1591,4516]$ the threshold interval of the integer m, we always have the counterexamples.

Remark 2.18. For the minimum dimension fixed d and that the amplitude of the m threshold interval increases in semigeometric progression of order k with $2 < q_k < 3$ one finds counterexample to the conjecture, however if $m > b_d$ then we have a verified conjecture.

Corollary 2.19. Let d be a appropriate dimension of any family of Ehrhart polynomials $g_{m,d,k}(n)$ according the conditions in (8) and (10), m a nonzero and nonnegative integer, $[a_d, b_d]$ a threshold interval, where a_d and b_d nonzero and nonnegative integers. If m belonging to the threshold interval $[a_d, b_d]$ then we have a counterexample to the conjecture of Beck and al. in [2]. Furthermore, for a pair dimension d if $m = a_d$ then this family of polynomials has a root β of real part $Re(\beta)$ superior to $d-1$.

Proof. It is immediately in accordance with the Proposition 2.15.

We give some examples where the conjecture of Beck and al. (12) is not verified for an odd dimension d.

Example 2.20. For $d_0 = 15$ and $m \in [9,173]$ we have

$g_{m,15,8}(n) = \prod_{j=8}^{15}(n+j) + m\prod_{j=0}^{7}(n-j)$ which have as roots α_0 with $Re(\alpha_0) > 14$. Then the appropriate threshold interval $[9,173]$ of m to find counterexamples to the conjecture for amplitude $\nu_0 = 173 - 9 + 1 = 165$.

In a similar way, for $d_1 = 17$ and $m \in [174, 621]$, the family of polynomials $g_{m,17,9}(n) = \prod_{j=9}^{17}(n+j) + m\prod_{j=0}^{8}(n-j)$ has a root α_1 such that $Re(\alpha_1) > 16$. This gives us an amplitude $\nu_1 = 621 - 174 + 1 = 448$.

Similarly, for $d_2 = 19$ and $m \in [622, 1882]$, we have

$g_{m,19,10}(n) = \prod_{j=10}^{19}(n+j) + m\prod_{j=0}^{9}(n-j)$ admit root α_2 with $Re(\alpha_2) > 18$. So its amplitude is $\nu_2 = 1882 - 622 + 1 = 1261$.

For $d_3 = 21$ and $m \in [1883,5295]$, $g_{m,21,11}(n)$ has a root α_3 with $Re(\alpha_3) > 20$. Its amplitude is $\nu_3 = 5295 - 1883 + 1 = 3413$.

....

For $d_8 = 31$ and $m \in [245658, 617927]$ then

$g_{m,31,16}(n) = \prod_{j=16}^{31}(n+j) + m\prod_{j=0}^{15}(n-j)$ and have a root

$\alpha_8 = 32.11838203 + 8.837919348i$. Hence $Re(\alpha_8) > 30$.

Lemma 2.21. Let d_i, with $0 \le i \le n$ the minimum dimensions for counterexamples of 12. Family $(d_i)_{0 \le i \le n}$ follows an arithmetic progression of the initial term $d_0 = 15$ and of reason $r = 2$.

Proof. It's immediate using the demonstration by recurrence.

We give some examples where the conjecture of Beck and al. (12) is not verified for odd dimension d.

Example 2.22. For $d_0 = 15$ et $m \in [9, 173]$ we have

$g_{m,15,8}(n) = \prod_{j=8}^{15}(n+j) + m\prod_{j=0}^{7}(n-j)$ whose root α_0 with $Re(\alpha_0) > 14$.

Then the appropriate threshold interval $[9,173]$ of m to find counterexamples to the conjecture for amplitude $\nu_0 = 173 - 9 + 1 = 165$.

Analogously, for $d_1 = 17$ and $m \in [174, 621]$, the family of polynomials

$g_{m,17,9}(n) = \prod_{j=9}^{17}(n+j) + m\prod_{j=0}^{8}(n-j)$ has root α_1 such that $Re(\alpha_1) > 16$. This give us an amplitude $\nu_1 = 621 - 174 + 1 = 448$.

Similarly, for $d_2 = 19$ with $m \in [622, 1882]$,

$g_{m,19,10}(n) = \prod_{j=10}^{19}(n+j) + m\prod_{j=0}^{9}(n-j)$ admit root α_2 with $Re(\alpha_2) > 18$. So its amplitude is $\nu_2 = 1882 - 622 + 1 = 1261$.

....

When $d_8 = 31$ and $m \in [245658, 617927]$ then

$g_{m,31,16}(n) = \prod_{j=16}^{31}(n+j) + +m\prod_{j=0}^{15}(n-j)$ have like root

$\alpha_8 = 32.11838203 + 8.837919348i$.

Hence $Re(\alpha_8) > 30$.

Lemma 2.23. Let d_i, with $0 \le i \le n$ the minimum dimensions for counterexamples of 12. The family $(d_i)_{0 \le i \le n}$ follows an arithmetic progression of initial term $d_0 = 15$ and reason $r = 2$.

Proof. This is immediate using the demonstration by recurrence.

Now, we denote by ν the amplitude of the threshold interval of nonnegative integer m satisfying $\nu = b_d - a_d + 1$. So to every threshold interval a corresponding amplitude is associated.

Proposition 2.24. The family of amplitudes $(\nu_i)_{1\le i\le n}$ follows a semigeometric progression of order k with $k<n$, of first term $\nu_1 = 448$ obtained for $a_d = 174$, $b_d = 621$ of corresponding dimension $d = 17$ and of reasons q_k between 2 and 3 (at constant decimal).
Proof. By applying the reasoning by recurrence one can get the result.

Proposition 2.25. The family of amplitudes $(\nu_i)_{1\le i\le n}$ defines a semigeometric sequence of order k with $k<n$, of first term $\nu_1 = 905$ obtained for $a_d = 423$, $b_d = 1327$, of corresponding dimension $d = 20$ and of reasons q_k included strict between 2 and 3.
Proof. Same reasoning as the Proposition 2.24.

Proposition 2.26. Let $(\nu_i)_{1\le i\le n}$ be a semigeometric sequence of order k, where $k<n$. From a certain rank $n \ge n_o$ the sequence $(\nu_i)_{1\le i\le n}$ is a geometric sequence.
Proof. Let $(\nu_i)_{1\le i\le n}$ be a semigeometric sequence of order k, where $k<n$. According to the Proposition 2.10, there is a special semigeometric sequence $(q_k)_{1\le r<k}$ which is also a semigeometric sequence of order r where $r<k$, that is $(q_k)_{1\le r<k}$ is constructed from $(\nu_i)_{1\le i\le n}$. By the Proposition 2.13, the sequence $(\nu_i)_{1\le i\le n}$ is stationary from a certain rank $n \ge n_0$. Therefore, the sequence $(\nu_i)_{1\le i\le n}$ is a geometric sequence from a certain rank $n \ge n_0$.

Theorem 2.27. Let $g_{m,d,k}$ (resp. $g_{m,d,k'}$) a family of Ehrhart polynomials of dimension d with $d \ge 15$ (resp. $d \ge 18$) for d is odd (resp. pair) and of k (resp. k') degree. For all special semigeometric convergent sequence $(q_i)_{1\le i\le n}$ of $g_{m,d,k}$ (resp. $g_{m,d,k'}$), there is a threshold interval of integer m containing counterexamples.
Proof. By Propositions 2.24, 2.25 and 2.26, we can verify that the special semigeometric sequence $(q_i)_i$ is strictly decreasing and bounded below by a finite number $l \ge 2$. So $(q_i)_i$ is a convergent sequence converges to 2. Then there exists a map φ_{q_i} which, each q_i in $]2,3[$ associates m that takes its values in the threshold interval $[a_d, b_d]$ with an appropriate dimension d such as $d \ge 15$ (resp. $d \ge 18$) if it is odd (resp. pair) where $a_d \ge 9$. In other words, there is always a threshold interval of integer m containing counterexamples for each $(q_i)_i$ that takes values in the interval $]2,3[$ for a given minimum dimension. Hence the result.

Example 2.28. For a suitable odd dimension d, we have the calculation results (mode 'worksheet') in Maple 13 code.

- $d_0 = 15, m \in [9, 173]$ so we obtain $\nu_0 = (173 - 9) + 1 = 165$
- $d_1 = 17, m \in [174, 621]$ then $\nu_1 = (621 - 174) + 1 = 448$
- $d_2 = 19, m \in [622, 1882]$ we find $\nu_2 = 1261$ so $q_1 = \frac{\nu_2}{\nu_1} = 2.814732142857143$.
- $d_3 = 21, m \in [1883, 5295]$ we have $\nu_3 = 3413$ and

$q_2 = \frac{\nu_3}{\nu_2} = 2.706582077716098$.
- $d_{18} = 51, m \in [1971766755, 4766545270]$ so $\nu_{18} = 2794778516$ thus

$q_{17} = \frac{\nu_{18}}{\nu_{17}} = 2.414437890452458$.

Example 2.29. When the suitable dimension d is pair, we have
- $d = 18, m \in [6, 515]$ we obtain $\nu_0 = 515 - 6 + 1 = 510$.
- $d = 20, m \in [516, 1590]$ we have $\nu_1 = 1075$.
- $d = 22, m \in [1591, 4516]$ the corresponding amplitude is $\nu_2 = 2926$ so we

have $q_1 = \frac{\nu_2}{\nu_1} = 2.721860465116279$.
- $d = 30, m \in [83636, 213153]$ we have $\nu_6 = 129518$ then

$q_5 = \frac{\nu_6}{\nu_5} = 2.525160359517264$. And so on.

Remark 2.30. For an appropriate pair dimension, we can have counterexamples outside the threshold interval. For example, for $d = 22$ and $m = 4$; of even when $d = 30$ and $m = 3$, etc ...

2.4. Special polynomial

Definition 2.31. We define p_s a special polynomial . $p_k(n)$ is a polynomial of degree k in variable n without constant term, obtained from constant term of a family of Ehrhart polynomials whose coefficients are the terms of the decomposition product of decreasing factors of the constant term of this family of polynomials. Then p_s is written by

$$p_s(n) = p_k(n) + a_0$$

where $p_k(n) = \sum_{i=1}^{k} C_{k,i} n^i, k \geq 1$ and $J_d \cong a_0 \; mod \; (d + k)$ cf. [1] and [9]. $C_{k,i}$ denote the coefficients of polynomial $p_k(n)$ determined by $C_{k,1} = d - k + 1, \ldots,$ and $C_{k,k} = d$.

Example 2.32. *For* $d = 5, k = 3$*, we have* $J_d = 3\times 4\times 5 = 60 = 5\times 4\times 3$ *with* $C_{3,1} = 3, C_{3,2} = 4, C_{3,3} = 5$ *then* $J_d = 60 \cong 4mod(8)$. *Thus*

$$p_k(n) = \sum_{i=1}^{3} C_{3,i}n^i = C_{3,1}n + C_{3,2}n^2 + C_{3,3}n^3 \text{ and}$$
$$p_s(n) = 5n^3 + 4n^2 + 3n + 4.$$

Proposition 2.33. Any family of Ehrhart polynomials has a unique special polynomial having the same dimension and degree that this family of polynomials.

Proof. Let $g_{m,d,k}$ (resp. $g_{m,d',k'}$) a family of Ehrhart polynomials verifying the conditions in (8) (resp. (10)). Its constant term can be decomposed into the product of decreasing factors, so because of the uniqueness of the decomposition of the constant term, the family of polynomials has a unique special polynomial of the same dimension and degree as $g_{m,d,k}$ or $g_{m,d',k'}$ according to the parity of its dimension.

Remark 2.34. If an integer $m \geq 2$ is up to the threshold interval $[a_d, b_d]$ compatible with the dimension d of the family of Ehrhart polynomials where we have counter examples to the conjecture of Beck, then there is one and only one Ehrhart polynomial which has the same dominant coefficient than its Special polynomial, that is, $m+1 = d$ (resp. $m+1 = d'$) for d (resp. d') is odd (resp. pair).

Proof. Immediate.

Proposition 2.35. All special polynomial p_s or p'_s satisfies the conjecture of Beck and al, for a minimum odd dimension d ($d \geq 17$) (resp. pair d' ($d' \geq 20$)).

Proof. In the Conjecture 2.1, if $m \in [174,621]$ and $d = 17$ we have a counterexample and the corresponding special polynomial has for dominant coefficient $d = m+1$, which gives us the value of $m = 16$ that does not belong to the threshold interval $[174,621]$. According to the Corollary 2.19 , the special polynomial verifies the conjecture of Beck and al. It is the same for $m \in [516,1590]$ and $d' = 20$, we obtain a counterexample, the fact that the dominant coefficient of special polynomial is $d = m+1$ then $m = 19$ which do not belong to the threshold interval $[516,1590]$ so it verifies the conjecture. Suppose that m belonging to the threshold interval $[a_d, b_d]$ and d is the dimension of the Ehrhart polynomial to have a counterexample then according the Proposition 2.33 there is a one special polynomial that has the same degree and dimension that the Ehrhart polynomial such as $d = m+1$, that is $m = d-1$ does not belong to $[a_d, b_d]$. So we have a contradiction. This completes the proof.

Let I_d an interval containing a family of Ehrhart polynomials of dimension d, for all $m \geq 2$; φ_d an application from I_d to $\mathbb{R}[n]$ such that

$$\varphi_d : I_d \to \mathbb{R}[n]$$

$$g \mapsto \varphi_d(g),$$

where g denote a family of Ehrhart polynomials of dimension d into I_d.

Proposition 2.36. The map φ_d is a diffeomorphism.

Proof. The map φ_d is bijective. Because, each family of polynomials g of I_d corresponds with one and only special polynomial p_s such as $\varphi_d(g) = p_s(n)$ and, p_s has one and only antecedent g. Being φ_d and φ_d^{-1} are C^∞ class then continues. Hence the proof.

REFERENCES

1. A.F.Beardon, Sums of powers of integers, *Amer. Math. Monthly.,* 103 (1996) 201213.
2 .M.Beck, J.A.De Loera, M. Develin, J. Pfeifle and R.P.Stanley, Coefficients and Roots of Ehrhart polynomials, *Contemp. Math.,* 374 (2005) 15-36.
3. E. Ehrhart, Démonstration de la loi de réciprocité pour un polyèdre entier, *C.R.Acad. Sci. Paris,* 265 (Série I) (1967) 5-7.
4.. Rajasingh, B.Rajan and V.Annamma, On total vertex irregularity strength of triangle related graphs, *Annals of Pure and Applied Mathematics,* 1**(**2) (2012) 108-116.
5 .M.G.Voskoglou, A triangular fuzzy model for assessing problem solving skills, *Annals of Pure and Applied Mathematics,* **7(**1) (2014) 53-58.
6. A.Higashitani, Counterexamples of the Conjecture on Roots of Ehrhart polynomials, *Discrete Comput Geom.,* 47 (2012) 618-623.
7. B.Braun and M.Develin, Ehrhart polynomial roots and Stanley's non-negativity theorem, *Contemp. Math .,* 452 (2008) 67-78.
8 .T.Matsui, A.Higashitani, Y.Nagazawa, H.Ohsugi and T.Hibi, Roots of Ehrhart polynomials arising from graphs, *J. Algebr. Comb.,* 34 (2011) 721-749.
9 .M.E.Mikkawy and F.Atlan, Notes on the power sum, *Annals of Pure and Applied Mathematics,* 9 (2) (2015) 215-232.
10. H.Ohsugi and K.Shibata, Smooth Fano polytopes whose Ehrhart polynomial has a root with large real part, *Discrete Comput Geom.,* 47 (2012) 624-628.