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Some Characterization of Neutral *n*-ideals and Distributive *n*-ideals of a Lattices

Md. Rakibul Hasan

Department of Computer Science & Engineering, Southeast University
Dhaka-1213, Bangladesh.
Email: md.rakibul_hasan@yahoo.com

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Abstract. Standard and neutral elements (ideals) of a lattice were studied by many authors. In this paper, the author has given some characterizations of n -ideals and extended some of the results. He also includes a characterization of neutral n -ideals of a lattice when n is a neutral element and including some results on distributive n-ideals of a lattices.

***Keywords*:** Neutral element, Standard n-congruence, Distributive n-ideals

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1. Introduction

Standard and neutral elements (ideals) in a lattice L were studied by G. Grätzer and Schmidt in [2] also see [1]. These concepts allow us to study a larger class of non-distributive lattices. Again in [4] and [5], Noor and Latif extended those concepts to study standard n-ideals in a lattice. In this paper I will examine some of the properties of standard and neutral n-ideals. I also discussed distributive n-ideals of lattices.

An element s of a lattice L is called neutral if

(i) $x \wedge (s \vee y) = (x \wedge s) \vee (x \wedge y)$ for all $x, y \in L$ and

(ii) for all $x, y \in L$, $s \wedge (x \vee y) = (s \wedge x) \vee (s \wedge y)$.

For a fixed element n of a lattice L, a convex sublattice containing n is called an n-ideal. The idea of n-ideals is a kind of generalization of both ideals and filters of lattices. The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set-inclusion. Moreover, $\{n\}$ and L are respectively the smallest and the largest elements of $I_n(L)$. For any two n-ideals I and J of L it is easy to check that $I \wedge J = I \cap J = \{x \in L : x = m(i, n, j)$ for some $i \in I,\ j \in J\}$, where $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $I \vee J = \{x \in L : i_1 \wedge j_1 \le x \le i_2 \vee j_2,$ for some $i_1, i_2 \in I$ and $j_1, j_2 \in J\}$. The n-ideal generated by a finite number of elements $a_1, a_2, ..., a_m$ is called a *finitely generated n-ideal* denoted by $< a_1, a_2, ..., a_m >_n$, which is the interval $[a_1 \wedge a_2 \wedge ... \wedge a_m \wedge n,\ a_1 \vee a_2 \vee ... \vee a_m \vee n >_n]$. The n-ideal generated by a single element a is called a principal n-ideal, denoted by $< a >_n = [a \wedge n,\ a \vee n]$. For detailed literature on n-ideals we refer the reader to consult [3].

An n-ideal of a lattice L is called neutral n*-ideal* of L if it is a neutral element of $I_n(L)$. The following characterization of neutral n-ideals is due to [4].

For any two n-ideals I and J of L, it is easy to check that $I \wedge J = I \cap J$ $= \{x \in L : x = m(i, n, j)$ for some $i \in I,\ j \in J\}$, where $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $I \vee J =$, $\{x \in L : i_1 \wedge j_1 \leq\ x \leq i_2 \vee j_2$ for $i_1,\ i_2 \in I$ and $j_1,\ j_2 \in J\}$.

The n-ideal generated by a finite numbers of elements $a_1, a_2, ..., a_m$ is called a finitely generated n-ideal, denoted by $< a_1, a_2, ..., a_m >_n$. Moreover, $< a_1, a_2, ..., a_m >_n$ is the interval $[a_1 \wedge a_2 \wedge .... \wedge a_m \wedge n, a_1 \vee a_2 \vee .... \vee a_m \vee n]$. The n-ideal generated by single element a is called a *principal* n*-ideal*, denoted by $< a >_n$ and $< a >_n = [a \wedge n,\ a \vee n]$. For detailed literature on n-ideals we refer the reader to consult [3,5].

Theorem 1.1. *Let* n *be a neutral element of a lattice* L. *An* n*-ideal* S *is a standard* n*-ideal if and only if for any* n*-ideal* K,

$S \vee K = \{x \in L : x = (x \wedge s_1) \vee (x \wedge k_1) \vee (x \wedge n)\}$

$= \{x \in L : x = (x \vee s_2) \wedge (x \vee k_2) \wedge (x \vee n)\}$ for some $s_1, s_2 \in S$ and $k_1, k_2 \in K$.

We start this paper with the following characterization of standard n-ideals.

Theorem 1.2. *Let* n *be a neutral element of a lattice* L, *An* n*-ideal S of a lattice* L *is standard if and only if* $< a >_n \cap (S \vee < b >_n) = (< a >_n \cap S) \vee (< a >_n \cap < b >_n)$ *for all* $a, b \in L$.

Proof: Suppose S is standard. Then obviously the above relation holds.
Conversely, suppose above relation holds for all a, b$\in$ L. Let K be an n-ideal of L and $x \in S \vee K$. Then $s_1 \wedge k_1 \leq x \leq s_2 \vee k_2$ for some $s_1, s_2 \in S$ and $k_1, k_2 \in K$. Now $n \leq x \vee n \leq s_2 \vee k_2 \vee n$ implies that $x \vee n \in < x >_n \cap (S \vee < k_2 \vee n >_n)$ $= (< x >_n \cap S) \vee (< x >_n \cap < k_2 \vee n >_n)$. Thus $x \vee n \leq t \vee r$ for some $t \in < x >_n \cap S$ and $r \in < x >_n \cap < k_2 \vee n >_n$. Then $t = (x \wedge s) \vee (x \wedge n) \vee (s \wedge n)$ for some $s \in S$ and $r \leq (x \vee n) \wedge (k_2 \vee n) = (x \wedge k_2) \vee n$, as n is neutral. Hence $x \vee n \leq (x \wedge s) \vee (x \wedge k_2) \vee n$, and so $x = x \wedge (x \vee n) \leq x \wedge ((x \wedge s) \vee (x \wedge k_2) \vee n) = (x \wedge s) \vee (x \wedge k_2) \vee (x \wedge n) \leq x$. Thus $x = (x \wedge s) \vee (x \wedge k_2) \vee (x \wedge n)$. By a dual proof of above we can prove that $x = (x \vee s') \wedge (x \vee k_1) \wedge (x \vee n)$ for some $s' \in S$. Therefore by Theorem 1.1, S is standard. □

An element $d \in L$ is called a dual distributive element if $d \wedge (x \vee y) = (d \wedge x) \vee (d \wedge y)$ for all $x, y \in L$. Hence an element which is both standard and dual distributive is a neutral element.

An n-ideal D is called a dual distributive n-ideal if it is a dual distributive element of $I_n(L)$. Now we give the following characterization of a dual distributive n-ideal.

Theorem 1.3. For $n \in L$, an n-ideal D is dual distributive if and only if $D \cap (<a>_n \vee <b>_n) = (D \cap <a>_n) \vee (D \cap <b>_n)$ for all $a, b \in L$.

Proof. If D is dual distributive, then clearly the relation holds.

Conversely, suppose the given relation holds for all $a, b \in L$. Suppose $I, J \in I_n(L)$. Let $x \in D \cap (I \vee J)$. Then $x \in D$ and $i_1 \wedge j_1 \leq x \leq i_2 \vee j_2$ for some $i_1, i_2 \in I, j_1, j_2 \in J$. Then $x \vee n \in D \cap (<i_2 \vee n>_n \vee <j_2 \vee n>_n)$ $= (D \cap <i_2 \vee n>_n) \vee (D \cap <j_2 \vee n>_n) \subseteq (D \cap I) \vee (D \cap J)$.

A dual proof also shows that $x \wedge n \in (D \cap I) \vee (D \cap J)$. Then by convexity of n-ideal $x \in (D \cap I) \vee (D \cap J)$. Therefore, $D \cap (I \vee J) \subseteq (D \cap I) \vee (D \cap J)$. Since the reverse inclusion is trivial, so D is dual distributive. □

2. Distributive *n*-ideal

An n-ideal I of a lattice L is called a *distributive n-ideal* if it is a distributive element of the lattice $I_n(L)$. That is, I is called distributive if for all $J, K \in I_n(L)$, $I \vee (J \cap K) = (I \vee J) \cap (I \vee K)$.

We start this section with the following characterization of distributive n-ideals.

Theorem 2.1. *An n-ideal I of a lattice L is distributive if and only if*

$$I \vee (<a>_n \cap <b>_n) = (I \vee <a>_n) \cap (I \vee <b>_n) \text{ for all } a, b \in L.$$

Proof: If I is distributive, then the condition clearly holds form the definition. To prove the converse, suppose given equation holds for all $a, b \in L$. Let J and K be any two n-ideals of L. Obviously $I \vee (J \cap K) \subseteq (I \vee J) \cap (I \vee K)$. To prove the reverse inclusion, let $x \in (I \vee J) \cap (I \vee K)$. Then $x \in I \vee J$ and $x \in I \vee K$. Then $i_1 \wedge j_1 \leq x \leq i_2 \vee j_2$ and $i_3 \wedge k_3 \leq x \leq i_4 \vee k_4$ for some $i_1, i_2, i_3, i_4 \in I$, $j_1, j_2 \in J$ and $k_3, k_4 \in K$. Now $n \leq x \vee n \leq i_2 \vee j_2 \vee n$ implies that $x \vee n \in I \vee <j_2 \vee n>_n$. Similarly $n \leq x \vee n \leq i_4 \vee k_4 \vee n$ implies that $x \vee n \in I \vee <k_4 \vee n>_n$. Thus, $x \vee n \in (I \vee <j_2 \vee n>_n) \cap (I \vee <k_4 \vee n>_n)$ $= I \vee (<j_2 \vee n>_n \cap <k_4 \vee n>_n) \subseteq I \vee (J \cap K)$. By a dual proof of above, we can show that $x \wedge n \in I \vee (J \cap K)$. Thus by convexity, $x \in I \vee (J \cap K)$. Therefore, $I \vee (J \cap K) = (I \vee J) \cap (I \vee K)$, and so I is distributive. □

Now we give another characterization of distributive n-ideal. To prove this we need the following lemma which is well known and is due to [1, Theorem-2, Page-139].

Lemma 2.2. *An element* a *of a lattice* L *is distributive if and only if the relation* θ_a *defined by* $x \equiv y\theta_a$ *if and only if* $x \vee a = y \vee a$ *is a congruence.* □

Theorem 2.3. *An* n*-ideal* I *of a lattice* L *is distributive if and only if the relation* $\Theta(I)$ *defined by* $x \equiv y\,\Theta(I)\ (x,\ y \in L)$ *if and only if* $x \vee i_1 = y \vee i_1$ *and* $x \wedge i_2 = y \wedge i_2$ *for some* $i_1, i_2 \in I$ *is the congruence generated by* I *.*

Proof: At first we shall show that $x \equiv y\,\Theta(I)$ if and only if $<x>_n \equiv <y>_n \Theta_I$ in $I_n(L)$. Let $x \equiv y\,\Theta(I)$. Then $x \vee i_1 = y \vee i_1$ and $x \wedge i_2 = y \wedge i_2$ for some $i_1, i_2 \in I$. Now $x \wedge i_2 = y \wedge i_2 \le y \le y \vee i_1 = x \vee i_1$ implies that $y \in <x>_n \vee I$. Similarly $x \in <y>_n \vee I$. Therefore, $<x>_n \vee I = <y>_n \vee I$, which implies that, in $I_n(L)$. Conversely, if $<x>_n \equiv <y>_n \Theta_I$ in $I_n(L)$, then $<x>_n \equiv <y>_n \Theta_I <x>_n \vee I = <y>_n \vee I$. Then $x \in <y>_n \vee I$ and so $y \wedge n \wedge i_1 \le x \le y \vee n \vee i_2$. Similarly, $x \wedge n \wedge i_3 \le y \le x \vee n \vee i_4$. Thus $x \le y \vee n \vee i_2 \le x \vee n \vee i_2 \vee i_4$ which implies $x \vee n \vee i_2 \vee i_4 = y \vee n \vee i_2 \vee i_4$. Similarly, $x \wedge n \wedge i_1 \wedge i_3 = y \wedge n \wedge i_1 \wedge i_3$. That is, $x \vee i = y \vee i$ and $x \wedge i' = y \wedge i'$ where $i = n \vee i_2 \vee i_4$ and $i' = n \wedge i_1 \wedge i_3$. Therefore, $x \equiv y\,\Theta(I)$.

Above proof shows that $\Theta(I)$ is a congruence in L if and only if Θ_I is a congruence in $I_n(L)$. But by Lemma 2.2, Θ_I is a congruence if and only if I is distributive in $I_n(L)$ and this completes the proof. □

By [1] we know that an element $n \in L$ is neutral if and only if for all $a, b \in L$, $(a \wedge b) \vee (a \wedge n) \vee (b \wedge n) = (a \vee b) \wedge (a \vee n) \wedge (b \vee n)$. Since this relation is self dual, so the dual condition of neutrality also implies the neutrality. So we have following extension of above theorem.

Theorem 2.4. For $a_1, a_2, ..., a_m, n \in L$, $<a_1, a_2, ..., a_m>_n$ is neutral if $a_1 \wedge n$, $a_2 \wedge n, ..., a_m \wedge n$ and $a_1 \vee n, a_2 \vee n, ..., a_m \vee n$ are all neutral elements in L.

Proof. Suppose $a_1 \wedge n, a_2 \wedge n, ..., a_m \wedge n$ and $a_1 \vee n, a_2 \vee n, ..., a_m \vee n$ are neutral. Then $a_1 \wedge a_2 \wedge ... \wedge a_m \wedge n$ and $a_1 \vee a_2 \vee ... \vee a_m \vee n$ are also neutral. By Theorem 1.4, $<a_1, a_2, ..., a_m>_n$ is standard. So we need to show only the dual distributive property. Let $I, J \in I_n(L)$ and $x \in <a_1, a_2, ..., a_m>_n \cap (I \vee J)$. Then $x \in <a_1, a_2, ..., a_m>_n$ and $i_1 \wedge j_1 \le x \le i_2 \vee j_2$ for some $i_1, i_2 \in I, j_1, j_2 \in J$. So

$x \vee n \leq (a_1 \vee a_2 \vee ... \vee a_m \vee n) \wedge [(i_2 \vee n) \vee (j_2 \vee n)] = [(a_1 \vee a_2 \vee ... \vee a_m \vee n) \wedge (i_2 \vee n)]$
$\vee [(a_1 \vee a_2 \vee ... \vee a_m \vee n) \wedge (j_2 \vee n)] \in (<a_1, a_2, ..., a_m>_n \cap I) \vee (<a_1, a_2, ..., a_m>_n \cap J)$

A dual proof shows that $x \wedge n \in (<a_1, a_2, ..., a_m>_n \cap I) \vee (<a_1, a_2, ..., a_m>_n \cap J)$

Hence by convexity $x \in (<a_1, a_2, ..., a_m>_n \cap I) \vee (<a_1, a_2, ..., a_m>_n \cap J)$

Thus

$<a_1, a_2, ..., a_m>_m \cap (I \vee J) \subseteq (<a_1, a_2, ..., a_m>_n \cap I) \vee (<a_1, a_2, ..., a_m>_n \cap J)$

Since the reverse inclusion is trivial, so

$<a_1, a_2, ..., a_m>_n \cap (I \vee J) = (<a_1, a_2, ..., a_m>_n \cap I) \vee (<a_1, a_2, ..., a_m>_n \cap J)$

Therefore, $<a_1, a_2, ..., a_m>_n$ is dual standard and so it is neutral. □

Following figure shows that the converse of above theorems are not true. Therefore $<a, f>_n = L$ isneutral in $I_n(L)$ but neither $a = a \vee n$ nor $f = f \vee n$ is even standard in L.

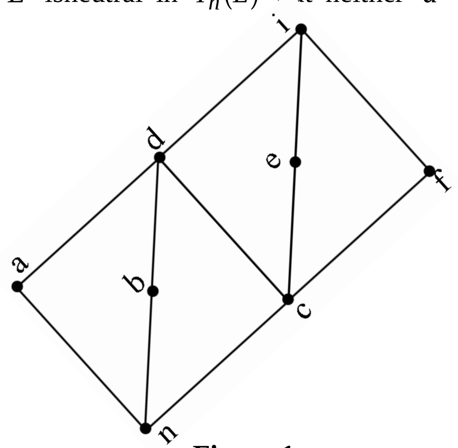

Figure 1:

Now we include a characterization of neutral n-ideals of a lattice with the help of principal n-ideals.

Theorem 2.5. *An n-ideal S of a lattice L is neutral if and only if* $(S \cap <a>_n) \vee (S \cap <b>_n) \vee (<a>_n \cap <b>_n) = (S \vee <a>_n) \cap (S \vee <b>_n) \cap (<a>_n \vee <b>_n)$ *for all $a, b \in L$.*

Proof. Let S be neutral. Then above relation holds as S is a neutral element of $I_n(L)$.

Now suppose the above relation holds for all $a, b \in L$. For any $I, J \in I_n(L)$, clearly $(S \cap I) \vee (S \cap J) \vee (I \cap J) \subseteq (S \vee I) \cap (S \vee J) \cap (I \vee J)$. To show the reverse inclusion, let $x \in (S \vee I) \cap (S \vee J) \cap (I \vee J)$. Then $x \leq s_1 \vee i_1, x \leq s_2 \vee j_2, x \leq i_3 \vee j_3$ for some $s_1, s_2 \in S; i_1, i_3 \in I; j_2, j_3 \in J$. This implies $x \vee n \in (S \vee <i_1 \vee i_3 \vee n>_n)$
$\cap (S \vee <j_2 \vee j_3 \vee n>_n) \cap (<i_1 \vee i_3 \vee n>_n \vee <j_2 \vee j_3 \vee n>_n) =$
$(S \cap <i_1 \vee i_3 \vee n>_n) \vee (S \cap <j_2 \vee j_3 \vee n>_n) \vee (<i_1 \vee i_3 \vee n>_n \cap <j_2 \vee j_3 \vee n>_n) \subseteq$

$(S \cap I) \vee (S \cap J) \vee (I \cap J)$ by using the given relation. A dual proof of above shows that $x \wedge n \in (S \cap I) \vee (S \cap J) \vee (I \cap J)$. Thus by convexity,
$x \in (S \cap I) \vee (S \cap J) \vee (I \cap J)$.Therefore $(S \cap I) \vee (S \cap J) \vee (I \cap J)$
$= (S \vee I) \cap (S \vee J) \cap (I \vee J)$. Hence by [1] S is a neutral n-ideal. $\square$

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