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Quadratic form of Subgroups of a finite Abelian p-Group of Rank Two

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Abstract. Formula for the number of subgroups of a finite abelian group of rank two is already determined. We can associate a quadratic from with finite abelian group of rank two. We prove this quadratic form is positive definite.

Keywords: subgroups, Euler's function, number of divisor, quadratic form, positive definite matrix.

AMS Mathematics Subject Classification (2010): 20K27, 15A63

1. Introduction

One of the famous problems in group theory is to find the number of subgroups of an abelian group. In [1], an explicit formula for the number of subgroups of a finite abelian group of rank two is indicated. The number f(m, n) distinct subgroups of group $Z_{p^m} \times Z_{p^n}$ with $m, n \in N \cup \{0\}$, can be associated with quadratic form. Our goal of our paper is to show that the quadratic form is positive definite

In [1], authors provides the formula for total number of subgroups of Group $Z_{p^m} \times Z_{p^n}$ are $\sum_{q|(p^m,p^n)} \tau\left(\frac{p^m}{q}\right) \tau\left(\frac{p^n}{q}\right) \varphi(q)$. For Simplification of $q|(p^m, p^n)$, one can say that q is some power of p (say p^r), then above result can be rewritten as $\sum_{r=0}^{\min\{m,n\}} \tau(p^{m-r})\, \tau(p^{n-r}) \varphi(p^r)$.

2. Main results

Let us consider the matrix $A_n = (a_{ij}) \in M_{n+1}(N)$ defined by $a_{ij} = f(i,j) \forall\ i, j = 0,1,2, \dots, n$. Clearly, A_n is symmetric and so it induces quadratic form $\sum_{i,j=0}^{n} a_{ij} x^i y^j$. Now we compute the principal minors in the top left corner of A_n for this we have to find an explicit expression for det (A_k) for all $k = 0,1,2, \dots, n$.

Theorem 1. For each $0 \leq k \leq n$, the following equality hold

$$\mathbf{det}(\boldsymbol{A_k}) = \prod_{\boldsymbol{i=0}}^{\boldsymbol{k}} \boldsymbol{\varphi}(\boldsymbol{p^k}).$$

Proof: Let $k \in \{0,1,2,..,n\}$ be fixed. Use of Theorem 1, the determinant $\det(A_k)$ is given by: $\det\left(\sum_{r=0}^{\min\{i,j\}} \tau(p^{i-r})\,\tau(p^{j-r})\varphi(p^r)\right)_{i,j=\overline{0,k}}$

Hence, for a fixed s$\in \{0,1,2,..,k\}$, the line L_s of A_k is the following form:

$L_s =$

$(\tau(p^s)\tau(p^0)\varphi(1) \quad \sum_{r=0}^{\min\{s,1\}} \tau(p^{s-r})\,\tau(p^{1-r})\varphi(p^r) \quad \sum_{r=0}^{\min\{s,2\}} \tau(p^{s-r})\,\tau(p^{2-r})\varphi(p^r)$
$\dots \; \dots \sum_{r=0}^{\min\{s,k\}} \tau(p^{s-r})\,\tau(p^{k-r})\varphi(p^r))$.

We shall apply transformations on the matrix A_k in order to put it into upper triangular matrix form. So, we consider consecutive transformations $L_t = L_t - (t+1)L_0$ for every $= \overline{1,k}$.

One obtains that

$L_0 = (\tau(p^0)\tau(p^0)\varphi(1) \;\; \tau(p^1)\tau(p^0)\varphi(1) \;\; \tau(p^2)\tau(p^0)\varphi(1) \dots \tau(p^k)\tau(p^0)\varphi(1))$

$L_s = (0 \;\; \tau(p^0)\tau(p^0)\varphi(p) \;\; \sum_{r=1}^{\min\{s,2\}} \tau(p^{s-r})\,\tau(p^{2-r})\varphi(p^r) \dots$
$\sum_{r=1}^{\min\{s,k\}} \tau(p^{s-r})\,\tau(p^{k-r})\varphi(p^r))$ $\forall$ s=1,2,…,k.

Now we apply consecutive transformations $L_t = L_t - (t)L_1$ for every $t = \overline{2,k}$.

One obtains that

$L_0 = (\tau(p^0)\tau(p^0)\varphi(1) \;\; \tau(p^1)\tau(p^0)\varphi(1) \;\; \tau(p^2)\tau(p^0)\varphi(1) \dots \tau(p^k)\tau(p^0)\varphi(1))$

$L_1 = (0 \;\; \tau(p^0)\tau(p^0)\varphi(p) \;\; \tau(p^1)\tau(p^0)\varphi(p) \dots \tau(p^{k-1})\tau(p^0)\varphi(p))$

$L_s = (0 \;\; 0 \;\; \tau(p^{s-2})\tau(p^0)\varphi(p^s) \dots$
$\sum_{r=2}^{\min\{s,k\}} \tau(p^{s-r})\,\tau(p^{k-r})\varphi(p^r))$ $\forall$ s=2,…,k.

So, after k steps of above algorithm continue this process, we get

$L_0 = (\tau(p^0)\tau(p^0)\varphi(1) \;\; \tau(p^1)\tau(p^0)\varphi(1) \;\; \tau(p^2)\tau(p^0)\varphi(1) \dots \tau(p^k)\tau(p^0)\varphi(1))$

$L_1 = (0 \;\; \tau(p^0)\tau(p^0)\varphi(p) \;\; \tau(p^1)\tau(p^0)\varphi(p) \dots \tau(p^{k-1})\tau(p^0)\varphi(p))$

$L_2 = (0 \;\; 0 \;\; \tau(p^0)\tau(p^0)\varphi(p^2) \dots \tau(p^{k-2})\tau(p^0)\varphi(p^2)\,)$

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$L_k = (0 \;\; 0 \;\; 0 \;\; \dots \;\; \tau(p^0)\tau(p^0)\varphi(p^k))$

Hence

$\det(A_k) =$

$$\begin{vmatrix} \tau(p^0)\tau(p^0)\varphi(1) & \tau(p^1)\tau(p^0)\varphi(1) & \tau(p^2)\tau(p^0)\varphi(1) & \dots & \tau(p^k)\tau(p^0)\varphi(1) \\ 0 & \tau(p^0)\tau(p^0)\varphi(p) & \tau(p^1)\tau(p^0)\varphi(p) & \dots & \tau(p^{k-1})\tau(p^0)\varphi(p) \\ 0 & 0 & \tau(p^0)\tau(p^0)\varphi(p^2) & \dots & \tau(p^{k-2})\tau(p^0)\varphi(p^2) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \tau(p^0)\tau(p^0)\varphi(p^k) \end{vmatrix}$$

$=\prod_{i=0}^{k} \varphi(p^k)$.

Now, the following two corollaries are obvious from Theorem 1.

Corollary 1. The quadratic form $\sum_{i,j=0}^{n} f(i,j)x^i y^j$ induced by the matrix A_k is positive definite, for all $k \in N$.

Corollary 2. For each $k \in N$, all eigenvalues of the matrix A_k are positive.

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