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Common Fixed Point Theorems for Weakly Compatible Mapping Satisfying Generalized Contraction Principle in Complete G-Metric Spaces

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Abstract. In this paper, we study some common fixed point results for weakly compatible mapping satisfying Generalized Contraction Principle in G-metric space by using a control function.

Keywords: Common fixed point, weakly compatible, generalized weak contraction, Altering distance function, control function.

AMS Mathematics Subject Classification (2010): 54H25

1. Introduction

Some generalizations of the notion of a metric space have been proposed by some authors. Gahler [1,2] coined the term of 2-metric spaces. This is extended to D-metric space by Dhage (1992) [3, 4]. Dhage proved many fixed point theorems in D-metric space. In 2006, Mustafa in collaboration with Sims introduced a new notion of generalized metric space called G-metric space [5]. In fact, Mustafa et al. studied many fixed point results for a self mapping in G-metric spaces under certain conditions; see [5, 6, 7, 8, 9].

2.Definitions and preliminaries

Definition 2.1. (Altering Distance Function [see 10]) A mapping $f: [0, \infty) \to [0, \infty)$ is called an Altering Distance Function if the following properties are satisfied.

(a) f is continuous and non-decreasing.
(b) $f(t) = 0$ if and only if $t = 0$.

Definition 2.2. (Control Function [see 10]) A Control Function ϕ is defined as $\varphi: R^+ \to R^+$ which is continuous at zero, monotonically increasing and $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.3. **[5]** Let X be a non empty set, and let $G: X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$,

$(G2)$ $G(x,x,y) > 0$ for all $x,y \in X$, with $x \neq y$.
$(G3)$ $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$, with $y \neq z$.
$(G4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$, (symmetry in all three variables)
$(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$ (rectangular inequality)
Then the function G is called a generalized metric, or more specially a G-metric on X, and the pair (X,G) is called a G-metric space.

Example 1.1. Let (X,d) be a usual metric space. Then (X,G_s) and (X,G_m) are G-metric spaces, where

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z) \text{ for all } x,y,z \in X$$

and

$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\} \text{ for all } x,y,z \in X.$$

Definition 2.4. [5] Let (X,G) and (X',G') be G-metric spaces and let $f:(X,G) \to (X',G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\varepsilon > 0$ there exist $\delta > 0$ such that $x,y \in X, G(a,x,y) < \delta$ implies that $G'(fa,fx,fy) < \varepsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Definition 2.5. [5] Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, therefore; we say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$; that is ,for any $\varepsilon > 0$, there exist N$\in N$ such that $G(x,x_n,x_m) < \varepsilon$ for all $n.m \geq$N.We call $x$is the limit of the sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.6. [5] Let (X,G) and (X',G') be G metric spaces, then a function $f: X \to X$ is said to be G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous, that is, whenever $\{x_n\}$ is G-convergent to x, $\{fx_n\}$ is G-convergent to $f(x)$.

Proposition 2.7. [5] Let (X,G) be a G-metric space. Then the following statements are equivalent

(a) $\{x_n\}$ is G-convergent to x.
(b) $G(x_n,x_n,x) \to 0$ as $n \to \infty$.
(c) $G(x_n,x,x) \to 0$ as $n \to \infty$.
(d) $G(x_n,x_m,x) \to 0$ as $n \to \infty$.

Proposition 2.8. [5] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-cauchy sequence if given $\varepsilon > 0$, there is N$\in N$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq$N; that is if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 2.9. [5] In a G-metric space (X,G), the following two statements are equivalent.

(1) The sequence $\{x_n\}$ is G-cauchy.
(2) For every $\varepsilon > 0$, there exist $N \in N$ such that $G(x_n,x_m,x_m) < \varepsilon$ for all $n,m \geq N$.

Definition 2.10. [5] A G-metric space (X, G) is said to be G-complete (or a complete G-metric pace) if every G-cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.11. [5] Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 2.12. [5] A G-metric space $(X, G)$is called a symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 2.13. [5] Every G-metric space (X, G) defines a metric space (X, d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

Note that, if (X, G) is a symmetric space G-metric space, then

$$d_G(x, y) = 2\, G(x, y, y) \text{ for all } x, y \in X$$

However, if (X, G) is not asymmetric space, then it holds by the G-metric properties that

$$\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y) \text{ for all } x, y \in X.$$

In general, these inequalities cannot be improved.

Proposition 2.14. [5] A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.

Proposition 2.15. [5] Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$, it follows that

(1) If $G(x, y, z) = 0$ then $x = y = z$.
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.
(3) $G(x, y, y) \leq 2\, G(y, x, x)$.
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.
(5) $G(x, y, z) \leq \frac{2}{3}\{G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 2.16. Two self maps T and f of a G-Metric Space (X, G) are said to be **weakly compatible** if $Tfx = fTx$ whenever $fx = Tx$ for all $x \in X$.

Definition 2.17. Let T and f be two self maps of a non empty subset M of a metric space X. The mapping T is called $\boldsymbol{f}$**-contraction** mapping, if there exist a real number $0 \leq k < 1$ such that $G(Tx, Ty, Tz) \leq k.G(fx, fy, fz)$ for all $x, y, z \in M$.

Definition 2.18. A mapping $T: X \to X$, where (X, G) is a G-metric space, is said to be a **Weak Contraction** if

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \emptyset\big(G(x, y, z)\big),$$

where $x, y, z \in X$ and $\emptyset: [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function such that $\emptyset(t) = 0$ if and only if $t = 0$.

Theorem 2.19. [11] Let (X, G) be a complete G- metric space and $T: X \to X$ be a mapping satisfying

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \emptyset\big(G(x, y, z)\big),$$

for all $x, y, z \in X$. If $\emptyset: [0, \infty) \to [0, \infty)$ is a continuous and non decreasing function with $\emptyset(t) = 0$ if and only if $t = 0$, then T has a unique fixed point in X.

Definition 2.20. A self mapping T of a metric space (X, G) is said to be **Weakly Contractive with respect to a self mapping $f: X \to X$** if for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq G(fx, fy, fz) - \emptyset(G(fx, fy, fz)).$$

where $\emptyset: [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\emptyset$ is positive on$(0, \infty)$, $\emptyset(0) = 0, \lim_{t\to\infty} \emptyset(t) = \infty$.

Note 2.1. If $= I$, the identity mapping, then the above definition is as follows.
A self mapping T of a metric space (X, G) is said to be Weakly Contractive with respect to a self mapping $f: X \to X$ if for all $x, y, z \in X$

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \emptyset\big(G(x, y, z)\big).$$

This is a **Weakly Contractive** Mapping.

Note 2.2. Combining the generalization of Contraction Principle and Weakly Contractive Mapping with respect to a self map in G-Metric Space we can obtain the following result.

Theorem 2.21. Let (X, G) be a complete G-Metric Space and a self map $T: X \to X$ be weakly contractive mapping with respect to a self mapping $f: X \to X$ if for all $x, y, z \in X$ and $T: X \to X$ is satisfying

$$\varphi\big(G(Tx, Ty, Tz)\big) \leq \varphi(G(fx, fy, fz)) - \emptyset(G(fx, fy, fz))$$

where $\emptyset: [0, \infty) \to [0, \infty), \varphi: [0, \infty) \to [0, \infty)$ are continuous and monotone non-decreasing functions with $\varphi(t) = 0 = \emptyset(t)$ if and only if $t = 0$, then T has a unique fixed point.

Theorem 2.22. [see 12] Let T and f be self maps of a G -metric space (X, G) satisfying

$$\varphi\big(d(Tx, Ty)\big) \leq \varphi\big(M(x, y)\big) - \emptyset(M(x, y)) \text{ for all } x, y, \in X$$

where $M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fy, Tx) + d(fx, Ty)]\}$ (1)
and $\emptyset, \varphi$:$[0, \infty) \to [0, \infty)$ are both continuous monotone non-decreasing functions with $\varphi(t) = 0 = \emptyset(t)$ if and only if $t = 0$. If TX is complete metric space and $TX \subset fX$, then T and f have coincidence point in X. Further, if T and f are weakly compatible, then they have a unique common fixed point in X.

Motivated by the above result, we address the same question on G-metric space for weakly compatible mappings satisfying a Generalized Contraction Principle condition given by (1), we establish a fixed point results in the third part of the paper. Our results are the following.

3. Main results

Theorem 3.1: let T and f be self maps of a complete G-metric space (X, G) satisfying

$$\varphi\big(G(Tx, Ty, Tz)\big) \leq \varphi\big(M(x, y, z)\big) - \emptyset\big(M(x, y, z)\big) \text{ for all } x, y, z \in X \quad (2)$$

where

$$M(x, y, z) = \max\{G(fx, fy, fz), G(fx, Tx, Tx), G(fy, Ty, Ty), G(fz, Tz, Tz),$$
$$\tfrac{1}{3}\big(G(fy, Tx, Tx) + G(fx, Ty, Ty)\big), \tfrac{1}{3}\big(G(fz, Ty, Ty) + G(fy, Tz, Tz)\big),$$

$$\left(G(fx,Tz,Tz)+G(fz,Tx,Tx)\right)\} \qquad (3)$$

and $\emptyset,\varphi:[0,\infty)\to[0,\infty)$ are both continuous monotone non-decreasing functions with $\varphi(t)=0=\emptyset(t)$ if and only if $t=0$. If TX is complete metric space and $TX\subset fX$, then T and f have coincidence point in X. Further, if T and f are weakly compatible, then they have a unique common fixed point in X.

Proof: let x_0 be an arbitrary point. Construct the sequence $\{x_n\}$ such that

$$fx_n=Tx_{n-1} \text{ for } n=1,2,3,\dots\dots\dots\dots \qquad (4)$$

this is possible since $TX\subset fX$.

Now

$$\varphi\left(G(Tx_n,Tx_{n+1},Tx_{n+1})\right)\le\varphi\left(M(x_n,x_{n+1},x_{n+1})\right)-\emptyset\left(M(x_n,x_{n+1},x_{n+1})\right). \qquad (5)$$

where

$$\begin{aligned}&M(x_n,x_{n+1},x_{n+1})\\&=\max\{G(fx_n,fx_{n+1},fx_{n+1}),G(fx_n,Tx_n,Tx_n),G(fx_{n+1},Tx_{n+1},Tx_{n+1}),\\&\quad G(fx_{n+1},Tx_{n+1},Tx_{n+1}),\tfrac{1}{3}\left(G(fx_{n+1},Tx_n,Tx_n)+G(fx_n,Tx_{n+1},Tx_{n+1})\right),\\&\quad \tfrac{1}{3}(G(fx_{n+1},Tx_{n+1},Tx_{n+1})+G(fx_{n+1},Tx_{n+1},Tx_{n+1})),\\&\quad \tfrac{1}{3}\left(G(fx_n,Tx_{n+1},Tx_{n+1})+G(fx_{n+1},Tx_n,Tx_n)\right)\}.\end{aligned}$$

$$\begin{aligned}&M(x_n,x_{n+1},x_{n+1})\\&=\max\{G(Tx_{n-1},Tx_n,Tx_n),G(Tx_{n-1},Tx_n,Tx_n),G(Tx_n,Tx_{n+1},Tx_{n+1}),\\&\quad G(Tx_n,Tx_{n+1},Tx_{n+1}),\tfrac{1}{3}\left(G(Tx_n,Tx_n,Tx_n)+G(Tx_{n-1},Tx_{n+1},Tx_{n+1})\right),\\&\quad \tfrac{1}{3}(G(Tx_n,Tx_{n+1},Tx_{n+1})+\left(G(Tx_n,Tx_{n+1},Tx_{n+1})\right),\\&\quad \tfrac{1}{3}(G(Tx_{n-1},Tx_{n+1},Tx_{n+1})+G(Tx_n,Tx_n,Tx_n))\}.\end{aligned}$$

$$\begin{aligned}&M(x_n,x_{n+1},x_{n+1})\\&=\max\{G(Tx_{n-1},Tx_n,Tx_n),G(Tx_n,Tx_{n+1},Tx_{n+1}),\tfrac{1}{3}G(Tx_{n-1},Tx_{n+1},Tx_{n+1}),\\&\quad \tfrac{2}{3}(G(Tx_n,Tx_{n+1},Tx_{n+1}),\tfrac{1}{3}(G(Tx_{n-1},Tx_{n+1},Tx_{n+1})\}.\end{aligned}$$

$$\begin{aligned}&M(x_n,x_{n+1},x_{n+1})\\&=\max\left\{G(Tx_{n-1},Tx_n,Tx_n),G(Tx_n,Tx_{n+1},Tx_{n+1}),\tfrac{1}{3}G(Tx_{n-1},Tx_{n+1},Tx_{n+1})\right\}.\end{aligned}$$

Therefore

$$M(x_n,x_{n+1},x_{n+1})=\max\{G(Tx_{n-1},Tx_n,Tx_n),G(Tx_n,Tx_{n+1},Tx_{n+1})\}.$$

Since, $\frac{1}{3}G(Tx_{n-1},Tx_{n+1},Tx_{n+1})\le\frac{1}{3}\{G(Tx_{n-1},Tx_n,Tx_n)+G(Tx_n,Tx_{n+1},Tx_{n+1})\}$.

$$\le\max\{G(Tx_{n-1},Tx_n,Tx_n),G(Tx_n,Tx_{n+1},Tx_{n+1})\}.$$

Therefore $M(x_n,x_{n+1},x_{n+1})=\max\{G(Tx_{n-1},Tx_n,Tx_n),G(Tx_n,Tx_{n+1},Tx_{n+1})\}$ (6)

From (5) and (6), we have

$$\begin{aligned}\varphi\left(G(Tx_n,Tx_{n+1},Tx_{n+1})\right)&\le\varphi(\max\{G(Tx_{n-1},Tx_n,Tx_n),G(Tx_n,Tx_{n+1},Tx_{n+1})\})\\&\quad-\emptyset(\max\{G(Tx_{n-1},Tx_n,Tx_n),G(Tx_n,Tx_{n+1},Tx_{n+1})\}).\end{aligned}$$

This implies

$$\varphi\left(G(Tx_n,Tx_{n+1},Tx_{n+1})\right)\le\varphi(G(Tx_{n-1},Tx_n,Tx_n)-\emptyset(G(Tx_{n-1},Tx_n,Tx_n)) \qquad (7)$$

$$\varphi\left(G(Tx_n,Tx_{n+1},Tx_{n+1})\right)\le\varphi(G(Tx_{n-1},Tx_n,Tx_n)$$

By monotone property of the function φ, we have

$$G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq G(Tx_{n-1}, Tx_n, Tx_n) \text{ for } n = 1,2,3 \ldots \ldots \ldots$$

Therefore the sequence $\{G(Tx_n, Tx_{n+1}, Tx_{n+1})\}$ is monotonic decreasing and continuous.

Therefore there exist a real number $r \geq 0$ such that

$$\lim_{n\to\infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = r \tag{8}$$

Taking $n \to \infty$ in equation (7), we get

$$\varphi(r) \leq \varphi(r) - \emptyset(r)$$

This is possible only when $r = 0$.

Therefore $\lim_{n\to\infty} G(Tx_n, Tx_{n+1}, Tx_{n+1}) = 0$ (9)

Next, we claim that $\{Tx_n\}$ is a Cauchy sequence.

Assume that $\{Tx_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and subsequences $\{n(i)\}, \{m(i)\}$ such that $m(i) < n(i) < m(i+1)$ along with

$$G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) \geq \varepsilon \text{ and } G(Tx_{m(i)}, Tx_{n(i)-1}, Tx_{n(i)-1}) < \varepsilon \tag{10}$$

Then it follows that

$$\begin{aligned}\varepsilon &\leq G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big)\\ &\leq G\big(Tx_{m(i)}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) + G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)})\end{aligned}$$

$$\varepsilon \leq G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big) \leq \varepsilon + G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big) \tag{11}$$

Let $i \to \infty$ and using (9) in (11)

$$\varepsilon \leq \lim_{i\to\infty} G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big) \leq \varepsilon + \lim_{i\to\infty} G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big)$$

$$\varepsilon \leq \lim_{i\to\infty} G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big) \leq \varepsilon + 0$$

$$\varepsilon \leq \lim_{i\to\infty} G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big) \leq \varepsilon$$

$$\text{Therefore } \lim_{i\to\infty} G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big) = \varepsilon \tag{12}$$

Now

$$\begin{aligned}&G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big)\\ &\leq G\big(Tx_{m(i)}, Tx_{m(i)-1}, Tx_{m(i)-1}\big) + G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big)\\ &\quad + G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big).\end{aligned}$$

$$\begin{aligned}G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big) &\leq 2\\ G\big(Tx_{m(i)-1}, Tx_{m(i)}, Tx_{m(i)}\big) &+ G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big)\\ &+ G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big).\end{aligned} \tag{13}$$

(Since $G(x, y, y) \leq 2G(y, x, x)$).

Letting $i \to \infty$ in (13)

$$\varepsilon \leq 2(0) + \lim_{i\to\infty} G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) + 0$$

$$\varepsilon \leq \lim_{i\to\infty} G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) \tag{14}$$

Again

$$G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) \leq G\big(Tx_{m(i)-1}, Tx_{m(i)}, Tx_{m(i)}\big)$$
$$+ G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) + G(Tx_{n(i)}, Tx_{n(i)-1}, Tx_{n(i)-1})$$

$$G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) \leq G\big(Tx_{m(i)-1}, Tx_{m(i)}, Tx_{m(i)}\big)$$
$$+ G(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}) + 2\, G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}) \tag{15}$$

(Since $G(x,y,y) \leq 2G(y,x,x)$)

Letting $i \to \infty$ in (15)

$$\lim_{i\to\infty} G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) \leq 0 + \varepsilon + 2(0)$$

$$\lim_{i\to\infty} G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) \leq \varepsilon \tag{16}$$

From (14) & (16)

$$\varepsilon \leq \lim_{i\to\infty} G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) \leq \varepsilon$$

$$\lim_{i\to\infty} G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big) = \tag{17}$$

Now using inequalities (2) and (10)

$$\varphi(\varepsilon) \leq \varphi\Big(G\big(Tx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big)\Big) \leq$$
$$\varphi(M\big(x_{m(i)}, x_{n(i)}, x_{n(i)}\big)) - \emptyset(M\big(x_{m(i)}, x_{n(i)}, x_{n(i)}\big)) \tag{18}$$

where

$$\begin{aligned} M(x_{m(i)}, x_{n(i)}, x_{n(i)}) = \max\, \{ & G\big(fx_{m(i)}, fx_{n(i)}, fx_{n(i)}\big), G\big(fx_{m(i)}, Tx_{m(i)}, Tx_{m(i)}\big), \\ & G\big(fx_{n(i)}, Tx_{n(i)}, Tx_{n(i)}\big), G(fx_{n(i)}, Tx_{n(i)}, Tx_{n(i)}), \\ & \tfrac{1}{3}\big\{G\big(fx_{n(i)}, Tx_{m(i)}, Tx_{m(i)}\big) + G\big(fx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big)\big\}, \\ & \tfrac{1}{3}\big\{G\big(fx_{n(i)}, Tx_{n(i)}, Tx_{n(i)}\big) + G\big(fx_{n(i)}, Tx_{n(i)}, Tx_{n(i)}\big)\big\}, \\ & \tfrac{1}{3}\big\{G\big(fx_{m(i)}, Tx_{n(i)}, Tx_{n(i)}\big) + G\big(fx_{n(i)}, Tx_{m(i)}, Tx_{m(i)}\big)\big\}\}. \end{aligned}$$

$$\begin{aligned} & M(x_{m(i)}, x_{n(i)}, x_{n(i)}) = \\ & \max\{G(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}), G(Tx_{m(i)-1}, Tx_{m(i)}, Tx_{m(i)}), \\ & G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big), G(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}), \\ & \tfrac{1}{3}\{G\big(Tx_{n(i)-1}, Tx_{m(i)}, Tx_{m(i)}\big) + G\big(Tx_{m(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big)\}, \\ & \tfrac{1}{3}\{G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big) + G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big)\}, \\ & \tfrac{1}{3}\{G\big(Tx_{m(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big) + G\big(Tx_{n(i)-1}, Tx_{m(i)}, Tx_{m(i)}\big)\}\}. \end{aligned}$$

$$\begin{aligned} & M(x_{m(i)}, x_{n(i)}, x_{n(i)}) = \\ & \max\{G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big), G(Tx_{m(i)-1}, Tx_{m(i)}, Tx_{m(i)}), \\ & G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big), \\ & \tfrac{1}{3}\{G\big(Tx_{n(i)-1}, Tx_{m(i)}, Tx_{m(i)}\big) + G\big(Tx_{m(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big)\}, \\ & \tfrac{2}{3}(G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big), \\ & \tfrac{1}{3}\{G\big(Tx_{n(i)-1}, Tx_{m(i)}, Tx_{m(i)}\big) + G\big(Tx_{m(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big)\}\}. \end{aligned}$$

$$\begin{aligned} & M(x_{m(i)}, x_{n(i)}, x_{n(i)}) = \\ & \max\{G\big(Tx_{m(i)-1}, Tx_{n(i)-1}, Tx_{n(i)-1}\big), G(Tx_{m(i)-1}, Tx_{m(i)}, Tx_{m(i)}), \\ & G\big(Tx_{n(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big), \\ & \tfrac{1}{3}\{G\big(Tx_{n(i)-1}, Tx_{m(i)}, Tx_{m(i)}\big) + G\big(Tx_{m(i)-1}, Tx_{n(i)}, Tx_{n(i)}\big)\}. \end{aligned}$$

Taking $i \to \infty$ on both sides in above equation, we obtain

$$\lim_{i\to\infty} M(x_{m(i)}, x_{n(i)}, x_{n(i)}) = \max\{\varepsilon, 0, 0, \leq \varepsilon\}$$

Therefore
$$\lim_{i\to\infty} M(x_{m(i)}, x_{n(i)}, x_{n(i)}) = \varepsilon \tag{19}$$

Letting $i \to \infty$ in (18) and using (19) in that, then we obtain

$$\varphi(\varepsilon) \le \varphi(\varepsilon) - \emptyset(\varepsilon)$$

which is a contradiction, has $\varepsilon > 0$. Thus $\{Tx_n\}$ is a Cauchy Sequence in TX which in turn implies that $\{fx_n\}$ is also Cauchy Sequence in X. Since TX is complete, $\{Tx_n\}$ converges to some $v \in TX$.

Since $TX \subset fX$ and $v = fu$ for some $u \in X$. thus $\{fx_n\}$ converges to fu.

Now

$$\lim_{n\to\infty} \varphi(G(Tx_n, Tu, Tu)) \le \lim_{n\to\infty} [\varphi(M(x_n, u, u) - \emptyset(M(x_n, u, u))]$$

where $\lim_{n\to\infty}(M(x_n, u, u) = \lim_{n\to\infty} \max\{G(fx_n, fu, fu),$

$G(fx_n, Tx_n, Tx_n), G(fu, Tu, Tu)\ G(fu, Tu, Tu), \frac{1}{3}\{G(fu, Tx_n, Tx_n) + G(fx_n, Tu, Tu)\},$

$\frac{1}{3}\{G(fu, Tu, Tu) + G(fu, Tu, Tu)\},\ \frac{1}{3}\{G(fx_n, Tu, Tu) + G(fu, Tx_n, Tx_n)\}\}.$

$M(x_n, u, u) = \max\ \{0,0, G(v, Tu, Tu), G(v, Tu, Tu), \frac{1}{3}\{G(v, v, v) + G(v, Tu, Tu)\},$

$\frac{1}{3}\{G(v, Tu, Tu) + G(v, Tu, Tu)\}, \frac{1}{3}\{G(v, Tu, Tu) + G(v, v, v)\}\}.$

$M(x_n, u, u) = \max\ \{0,0, G(v, Tu, Tu), G(v, Tu, Tu), \frac{1}{3}\{G(v, Tu, Tu)\},$

$$\frac{2}{3}\{G(v, Tu, Tu)\}, \frac{1}{3}\{G(v, Tu, Tu)\}.$$

Therefore $M(x_n, u, u) = G(v, Tu, Tu)$.

By monotone increasing property of φ & $\emptyset$, we have

$$\varphi\big(G(v, Tu, Tu)\big) \le \varphi\big(G(v, Tu, Tu)\big) - \emptyset(G(v, Tu, Tu)$$

which is possible only when$G(v, Tu, Tu) = 0$.

Thus $v = Tu = fu$ and u is the coincidence point of $T$and$f$.

Since $T$and $f$are weekly compatible, they commute at their coincidence point.

Hence $Tfu = fTu$ which implies $Tv = fv$. (20)

Now

$$\varphi(G(Tu, Tv, Tv) \le \varphi\big(M(u, v, v)\big) - \emptyset(M(u, v, v)), \tag{21}$$

where

$M(u, v, v) = \max\ \{G(fu, fv, fv), G(fu, Tu, Tu), G(fv, Tv, Tv), G(fv, Tv, Tv),$

$\frac{1}{3}\{G(fv, Tu, Tu) + G(fu, Tv, Tv)\},$

$\frac{1}{3}\{G(fv, Tv, Tv) + G(fv, Tv, Tv)\},$

$\frac{1}{3}\{G(fu, Tv, Tv) + G(fv, Tu, Tu)\}\}.$

$M(u, v, v)= \max\ \{G(v, Tv, Tv), G(v, v, v), G(Tv, Tv, Tv), G(Tv, Tv, Tv),$

$\frac{1}{3}\{G(Tv, v, v) + G(v, Tv, Tv)\},\ \frac{1}{3}\{G(Tv, Tv, Tv) + G(Tv, Tv, Tv)\},$

$\frac{1}{3}\{G(v, Tv, Tv) + G(Tv, v, v)\}\}.$

$M(u, v, v)=\max\ \{G(v, Tv, Tv), 0,0,0, \frac{1}{3}\{G(Tv, v, v) + G(v, Tv, Tv)\}, 0,$

$\frac{1}{3}\{G(v, Tv, Tv) + G(Tv, v, v)\}\}.$

$$M(u,v,v) = \max\{G(v,Tv,Tv), \frac{1}{3}\{G(Tv,v,v)+G(v,Tv,Tv)\}\}$$

$$M(u,v,v) = G(v,Tv,Tv). \tag{22}$$

Since $\frac{1}{3}\{G(v,Tv,Tv)+G(Tv,v,v)\} \leq \frac{1}{3}\{2G(v,Tv,Tv)+G(v,Tv,Tv)\}$

$$\frac{1}{3}\{G(v,Tv,Tv)+G(Tv,v,v)\} \leq G(v,Tv,Tv)\}.$$

Hence by using (22) in (21), we get

$$\varphi(G(v,Tv,Tv) = \varphi G(Tu,Tv,Tv) \leq \varphi(G(v,Tv,Tv)) - \emptyset(G(v,Tv,Tv))$$

$$\varphi(G(v,Tv,Tv) \leq \varphi(G(v,Tv,Tv)) - \emptyset(G(v,Tv,Tv))$$

This implies $\quad \emptyset(G(v,Tv,Tv) \leq 0$

which is possible only when $G(v,Tv,Tv) = 0$.

Therefore $\ v = Tv$.

Thus $v = Tv = fv$. (from (20))

Hence v is the common fixed point of T and f.

Uniqueness:

Let v and w be two fixed points of T and f.

That is $v = Tv = fv$ and $w = Tw = fw$.

By using inequality (4), we have

$$\varphi(G(Tv,Tw,Tw) \leq \varphi\big(M(v,w,w)\big) - \emptyset(M(v,w,w)) \tag{23}$$

where

$M(v,w,w) = \max\{G(fv,fw,fw), G(fv,Tv,Tv), G(fw,Tw,Tw), G(fw,Tw,Tw),$
$\frac{1}{3}\{G(fw,Tv,Tv)+G(fv,Tw,Tw)\}, \frac{1}{3}\{G(fw,Tw,Tw)+G(fw,Tw,Tw)\},$
$\frac{1}{3}\{G(fv,Tw,Tw)+G(fw,Tv,Tv)\}\}.$

$M(v,w,w) = \max\{G(v,w,w), G(v,v,v), G(w,w,w), G(w,w,w), \frac{1}{3}\{G(w,v,v)$
$+G(v,w,w)\}, \frac{1}{3}\{G(w,w,w)+G(w,w,w)\}, \frac{1}{3}\{G(v,w,w)+G(w,v,v)\}\}.$

$M(v,w,w) = \max\{G(v,w,w), 0,0,0, \{\frac{1}{3}\{G(w,v,v)+G(v,w,w)\}, 0\}.$

$M(v,w,w) = \max\{G(v,w,w), \{\frac{1}{3}\{G(w,v,v)+G(v,w,w)\}\}.$

$$M(v,w,w) = G(v,w,w). \tag{24}$$

Since $\{\frac{1}{3}\{G(w,v,v)+G(v,w,w)\} \leq \frac{1}{3}\{2G(v,w,w)+G(v,w,w)\}$

$$\frac{1}{3}\{G(w,v,v)+G(v,w,w)\} \leq G(v,w,w)\}.$$

Hence by using (24) in (23), we get

$\varphi(v,w,w) = \varphi(G(Tv,Tw,Tw) \leq \varphi\big(G(v,w,w)\big) - \emptyset(G(v,w,w))$

$$\varphi(v,w,w) \leq \varphi\big(G(v,w,w)\big) - \emptyset(G(v,w,w))$$

$$\emptyset(G(v,w,w) \leq 0$$

This is possible only when $G(v,w,w) = 0$.

Therefore $\quad v = w$

This proves the uniqueness of the common fixed point of T and f.

Example 3.1. Let $X = [0,1]$ and $d(x,y) = |x-y|$. Define

$G(x,y,z) = |x-y| + |y-z| + |z-x|$, then (X,G) is a complete G-metric space. Consider two self mappings T and f of X by $Tx = \frac{x}{2}$ and $fx = x$ for all $x \in X$.

Let $\varphi: [0,\infty) \to [0,\infty)$ be defined by

$$\varphi(t) = \begin{cases} t + \frac{t^2}{2} & if\ \ 0 \le t \le 1 \\ 0 & if\ t > 1 \end{cases} \tag{25}$$

and $\emptyset: [0,\infty) \to [0,\infty)$ defined by

$$\emptyset(t) = \begin{cases} \frac{3t^2}{8} & if\ \ 0 \le t \le 1 \\ 0 & if\ t > 1 \end{cases} \tag{26}$$

Now to verify inequality (2), LHS of (2)

$$\varphi\big(G(Tx,Ty,Tz)\big) = \varphi(|Tx - Ty| + |Ty - Tz| + |Tz - Tx|)$$

$$\varphi\big(G(Tx,Ty,Tz)\big) = \varphi\left(\left|\frac{x}{2} - \frac{y}{2}\right| + \left|\frac{y}{2} - \frac{z}{2}\right| + \left|\frac{z}{2} - \frac{x}{2}\right|\right),$$

$$\varphi\big(G(Tx,Ty,Tz)\big) = \varphi\left(\frac{|x-y|}{2} + \frac{|y-z|}{2} + \frac{|z-x|}{2}\right),$$

$$\varphi\big(G(Tx,Ty,Tz)\big) = \varphi\left(\frac{|x-y|+|y-z|+|z-x|}{2}\right),$$

$$\varphi\big(G(Tx,Ty,Tz)\big) = \varphi\left(\frac{G(x,y,z)}{2}\right),$$

$$\varphi\big(G(Tx,Ty,Tz)\big) = \frac{G(x,y,z)}{2} + \frac{\big(G(x,y,z)\big)^2}{8}. \tag{27}$$

Now to verify inequality (2), RHS of (2) is $\varphi\big(M(x,y,z)\big) - \emptyset\big(M(x,y,z)\big)$, (28)

where

$$M(x,y,z) = \max\{G(fx,fy,fz), G(fx,Tx,Tx), G(fy,Ty,Ty), G(fz,Tz,Tz),$$

$$\frac{1}{3}\big(G(fy,Tx,Tx) + G(fx,Ty,Ty)\big), \frac{1}{3}\big(G(fz,Ty,Ty) + G(fy,Tz,Tz)\big),\ \frac{1}{3}\big(G(fx,Tz,Tz) + G(fz,Tx,Tx)\big)\}$$

$$M(x,y,z) = \max\{G(x,y,z), G\left(x,\frac{x}{2},\frac{x}{2}\right), G\left(y,\frac{y}{2},\frac{y}{2}\right), G\left(z,\frac{z}{2},\frac{z}{2}\right),\ \frac{1}{3}\left(G\left(y,\frac{x}{2},\frac{x}{2}\right) + G\left(x,\frac{y}{2},\frac{y}{2}\right)\right)$$

$$\frac{1}{3}\left(G\left(z,\frac{y}{2},\frac{y}{2}\right) + G\left(y,\frac{z}{2},\frac{z}{2}\right)\right), \frac{1}{3}\left(G\left(x,\frac{z}{2},\frac{z}{2}\right) + G\left(z,\frac{x}{2},\frac{x}{2}\right)\right)\}.$$

$$M(x,y,z) = \max\{|x-y| + |y-z| + |z-x|, |x|, |y|, |z|, \frac{2}{3}\left(\left|y - \frac{x}{2}\right| + \left|x - \frac{y}{2}\right|\right), \frac{2}{3}\left(\left|z - \frac{y}{2}\right| + \left|y - \frac{z}{2}\right|\right), \frac{2}{3}\left(\left|x - \frac{z}{2}\right| + \left|z - \frac{x}{2}\right|\right)\}.$$

$$M(x,y,z) = |x-y| + |y-z| + |z-x| \text{ for all } x,y,z \in X,$$

$$M(x,y,z) = G(x,y,z) \text{ for all } x,y,z \in X \tag{29}$$

Substitute (29) in (28), we obtain RHS of (2) is

$$\varphi(G(x,y,z)) - \emptyset(G(x,y,z)),$$

From (24) and (25), we obtain RHS of (2) is

$$G(x,y,z) + \frac{\big(G(x,y,z)\big)^2}{2} - \frac{3\big(G(x,y,z)\big)^2}{8},$$

$$\text{RHS of (2) is} = G(x,y,z) + \frac{\big(G(x,y,z)\big)^2}{8} \tag{30}$$

From (26) and (29), we obtain

$$\frac{G(x,y,z)}{2} + \frac{\big(G(x,y,z)\big)^2}{8} \le G(x,y,z) + \frac{\big(G(x,y,z)\big)^2}{8},$$

This implies LHS ≤ RHS and inequality (2) is verified. Now, it is easy to see that $TX = \left[0, \frac{1}{2}\right) \subset fX = [0,1]$. Moreover, T and f are weakly compatible in X. Hence all the conditions of theorem 3.1 are satisfied. It may be noted that **0** is unique common fixed point of T and f.

Theorem 3.2. let T and f be self maps of a G-metric space (X, G) satisfying

$$\varphi\big(G(Tx, Ty, Tz)\big) \le k\ \varphi\big(M(x, y, z)\big) \text{ for all } x, y, z \in X \tag{31}$$

where

$$M(x, y, z) = \max\{G(fx, fy, fz), G(fx, Tx, Tx), G(fy, Ty, Ty), G(fz, Tz, Tz), \frac{1}{3}\big(G(fy, Tx, Tx) + G(fx, Ty, Ty)\big), \frac{1}{3}\big(G(fz, Ty, Ty) + G(fy, Tz, Tz)\big), \frac{1}{3}\big(G(fx, Tz, Tz) + G(fz, Tx, Tx)\big)\} \tag{32}$$

and φ:$[0, \infty) \to [0, \infty)$ is continuous monotone non-decreasing function with $\varphi(t) = 0$ if and only if $t = 0$. If TX is complete metric space and $TX \subset fX$, then T and f have coincidence point in X. Further, if T and f are weakly compatible, then they have a unique common fixed point in X.

Proof: By taking $\emptyset(t) = (1 - k)\ \varphi(t)$ in theorem 3.1 then condition (2) reduced to the condition (32), and the proof follows the theorem (3.1).

Theorem 3.3. let T and f be self maps of a G-metric space (X, G) satisfying

$$G(Tx, Ty, Tz) \le G(fx, fy, fz) - \emptyset\big(G(fx, fy, fz)\big) \text{ for all } x, y, z \in X \tag{33}$$

and $\emptyset$:$[0, \infty) \to [0, \infty)$ is continuous monotone non-decreasing function with $\emptyset(t) = 0$ if and only if $t = 0$. If TX is complete metric space and $TX \subset fX$, then T and f have coincidence point in X. Further, if T and f are weakly compatible, then they have a unique common fixed point in X.

Proof: By taking $\varphi(t) = t$ and $M(x, y, z) = G(fx, fy, fz)$ in theorem 3.1, then condition (2) reduced to the condition (33), and the proof follows the theorem (3.1).

Theorem 3.4. Let (X, G) be a complete G- metric space and $T: X \to X$ be a mapping satisfying

$$G(Tx, Ty, Tz) \le G(x, y, z) - \emptyset\big(G(x, y, z)\big), \tag{34}$$

for all$x, y, z \in X$. If $\emptyset: [0, \infty) \to [0, \infty)$ is a continuous and non decreasing function with $\emptyset(t) = 0$ if and only if $t = 0$, then T has a unique fixed point in X.

Proof: By taking $\varphi(t) = t$, $M(x, y, z) = G(fx, fy, fz)$ and $f = I$,(the identity function) in theorem 3.1, then condition (2) reduced to the condition (34), and the proof follows the theorem (3.1).

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