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On Some Complex Exact Solutions for Charap's Equations in a Generalized Form

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Abstract. Non linear differential equations in real variables generally present real solutions. In present paper the authors report complex solutions, obtained for two different sets of equations, namely Yang's R-Gauge equations in real form and a generalization of the Charap's Chiral equations. The former was reported previously by Chakraborty and Chanda in the year 2006 while the later case is located and reported by the present authors. The solutions for the two situations have been compared.

Keywords: nonlinear partial differential equations, complex solutions, particles and field

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1. Introduction

Non-linear differential equations are gradually attracting more and more importance these days. The first of the major reasons is, natural manifestations are non-linear in nature. Inspite of that scientists study linear approximations of those nonlinear equations because of the fact that it is very difficult to manage these nonlinear equations. However, recently there has been considerable development in this direction and now it is much easier to handle them. This is the second reason why the non-linear differential equations have drawn the attention of all the corners of scientific community [1,2,3]. Non-linear differential equations in real variables generally present real solutions. But in some situations one confronts with complex solutions. Normally people ignore such complex solutions. Also, there are examples where complex solution contribute to the physical understanding. One of the most celebrated example is the solutions of the Scrodinger equation. In order to avoid the absence of physical meaning of the imaginary terms in Scrodinger's wave function the celebrated Born explanation [4] came into rescue with the concept of the probability of a single electron in the hydrogen atom at every point and at every instant which is proportional to the probability density $|\hat{\psi}|^2 = \hat{\psi}\hat{\psi}^*$. Two important field equations, namely, the Yangs' R- Gauge equations [5] and the Charap's Chiral field equations [6] have real exact solutions which were reported by various authors [7,8,9,10,11,12] . In the present paper the authors report complex solutions for a generalization of the Charap's Chiral equations and compare them with those reported previously by Chakraborty and Chanda [7] for the Yangs' equations in real form. The comparison is worthwhile because of the fact that the two sets of equations are having several common features reported previously by Chakraborty and Chanda [7], Chanda, Ray and De [12] and Saha and Chanda [13]. At present it has not been possible to assign any physical interpretation. However, the physical situation where from the equations generate are important. And, the solutions reported here may attract some relevance in future. A few words about the similarity of the Yang's equations and Charap's equations are as follows. First, when written in terms of real variables the two sets of equations look similar in form. Second, both of the two sets of equations allow (i) reduction to equations in two independent variables which are conformally invariant equations permitting one to obtain infinitely many other solutions from any solution of these conformally invariant equations, and (ii) those reduced equations closely resemble to generalized Lund-Regge equations [14,15] given by

$$\theta_{11} + \theta_{22} - 2g(\theta) + h(\theta)(\lambda_1^2 + \lambda_2^2) = 0 \tag{1a}$$

$$[\lambda_1 \exp(-\int p(\theta)d\theta]_1 + [\lambda_2 \exp(-\int p(\theta)d\theta]_2 = 0 \tag{1b}$$

where $\theta = \theta(x^1, x^2), \lambda = \lambda(x^1, x^2), \theta_1 = \frac{\partial\theta}{\partial x^1}$ and so on.
With g=0, the equations reduce to a conformally invariant set of equations, a particular example of which is the physically interesting equations of two dimensional Hiesenberg ferromagnets [16,17].

2. The equations under study

The generalized form of Charap's equation used here has been reported for the first time in the work of Saha and Chanda [13].

$$\Box' \phi = k'' \eta'^{\mu\nu} \frac{\partial\phi}{\partial x^\mu}\frac{\partial\beta}{\partial x^\nu} \tag{2.1a}$$

$$\Box' \psi = k'' \eta'^{\mu\nu} \frac{\partial\psi}{\partial x^\mu}\frac{\partial\beta}{\partial x^\nu} \tag{2.1b}$$

$$\Box' \chi = k'' \eta'^{\mu\nu} \frac{\partial\chi}{\partial x^\mu}\frac{\partial\beta}{\partial x^\nu} \tag{2.1c}$$

where

$$\Box' \phi = \phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44}$$

$$\phi_1 = \frac{\partial\phi}{\partial x^1}, \phi_{11} = \frac{\partial^2\phi}{\partial x^{1^2}}$$

where

$\eta'^{\mu\nu} = 0 \; for \; \mu \neq \nu$

$= 1\ for\ \mu = \nu \neq 4$
$= \epsilon\ for\ \mu = \nu = 4$
$\epsilon = +1\ or -1$
$k'' = arbitrary\ constant.$
$\beta = \ln\left(f_\pi^2 + \phi^2 + \psi^2 + \chi^2\right)$
$f_\pi = constant$

Equations (2.1) can be written explicitly as

$$\phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44} = k''(\phi_1\beta_1 + \phi_2\beta_2 + \phi_3\beta_3 + \varepsilon\phi_4\beta_4) \tag{2.2a}$$

$$\psi_{11} + \psi_{22} + \psi_{33} + \varepsilon\psi_{44} = k''(\psi_1\beta_1 + \psi_2\beta_2 + \psi_3\beta_3 + \varepsilon\psi_4\beta_4) \tag{2.2b}$$

$$\chi_{11} + \chi_{22} + \chi_{33} + \varepsilon\chi_{44} = k''(\chi_1\beta_1 + \chi_2\beta_2 + \chi_3\beta_3 + \varepsilon\chi_4\beta_4) \tag{2.2c}$$

where $\beta = \ln\left(f_\pi^2 + \phi^2 + \psi^2 + \chi^2\right)$
$f_\pi = constant$

The equations (2.1) and (2.2) with $k'' = 1$ and $\epsilon = -1$ represent the celebrated Charap's equation. The equation were first written by Charap to describe Chiral field [6].

$For\ k'' = \frac{3}{2}, \epsilon = \pm 1$

Solutions of (2.2) as obtained by present authors are given by

$$\phi = \int (f_\pi^2 + \phi^2 + \psi^2 + \chi^2)^{k''} dX,\ \ \phi = \phi(X),\ \ \alpha = \alpha(X) \tag{2.3a}$$

$$\alpha_{\phi\phi} = \frac{A^2}{\alpha^3} + \frac{B^2\alpha}{\left(f_\pi^2 + \phi^2 + \alpha^2\right)^{2k''}} \tag{2.3b}$$

$$\psi = \alpha\cos\theta \tag{2.3c}$$

$$\chi = \alpha\sin\theta \tag{2.3d}$$

where

$$\theta = A\int\left(\frac{\left(f_\pi^2 + \phi^2 + \alpha^2\right)^{k''}}{\alpha^2}\right)dX + BY + C \tag{2.3e}$$

$A, B\ and\ C$ are constants of integration which are again functions of $(x^3 - x^4)$ and

$$X_{11} + X_{22} = 0 \tag{2.3f}$$

$$Y_{11} + Y_{22} = 0 \tag{2.3g}$$

$$X_1 = Y_2 \tag{2.3h}$$

$$X_2 = -Y_1 \tag{2.3i}$$

i.e. $X\ and\ Y$ are mutually conjugate Laplace solutions.

The procedure for obtaining (2.3) is the same as that used from the equation (5) to the equation (17) of the work of Chanda, Ray and De [12].

For $k^{''}=\frac{3}{2},\ A\neq 0, B\neq 0$

$$\left(\alpha_{\phi}^{2}\right)_{\alpha}=\frac{2A^{2}}{\alpha^{3}}+\frac{B^{2}}{4\alpha^{5}}$$

where $f_{\pi}^{2}+\phi^{2}=\alpha^{2}$

Integrating we get,

$$\alpha_{\phi}=\frac{\sqrt{16\alpha^{4}D-16\alpha^{2}A^{2}-B^{2}}}{4\alpha^{2}}$$

where D is an arbitrary constant

$$\phi_{X}\ \frac{dX}{d\alpha}=\frac{4\alpha^{2}}{\sqrt{16\alpha^{4}D-16\alpha^{2}A^{2}-B^{2}}}$$

Integrating both sides, we get

$$X=\frac{4}{2^{3/2}}\int\frac{d\alpha}{\alpha\sqrt{16\alpha^{4}D-16\alpha^{2}A^{2}-B^{2}}}$$

After integration we get

$$X=-\frac{2}{2^{3/2}B}\sin^{-1}\left[\frac{\frac{1}{\alpha^{2}}+\frac{8A^{2}}{B^{2}}}{\sqrt{\left(\frac{8A^{2}}{B^{2}}\right)^{2}+\frac{16D}{B^{2}}}}\right]+E$$

where E is an arbitrary constant

Let $F=\sqrt{\left(\frac{8A^{2}}{B^{2}}\right)^{2}+\frac{16D}{B^{2}}}$

$$G=\frac{8A^{2}}{B^{2}}$$

Taking $\frac{G}{F}=-1;$ we get $D=0$ and $F=-\frac{8A^{2}}{B^{2}}$

Under these assumptions we get,

$$\alpha=\frac{1}{\sqrt{2F}}\operatorname{cos}ec\left[\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right]$$

$$\theta = i\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right| + BY + C$$

Now $\phi = \int \left(f_\pi^2 + \phi^2 + \alpha^2 \right)^{k''} dX$

We have $k'' = \frac{3}{2}$, $f_\pi^2 + \phi^2 = \alpha^2$

So $\phi = 2^{3/2}\int \alpha^3 dX$

Substituting the value of α and integrating we get,

$$\phi = \frac{B^2}{16iA^3}\left[-\mathrm{cosec}\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\}\cot\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\} + \log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right]$$

Thus we get

$$\phi = -\frac{B^2 i}{16A^3}\left[-\mathrm{cosec}\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\}\cot\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\} + \log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right]$$

$$\psi = \frac{B}{4A}\mathrm{cosec}\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\}[-\sinh\left\{\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right\}\sin(BY+C)$$

$$-\ i\cosh\left\{\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right\}\cos(BY+C)]$$

$$\chi = \frac{B}{4A}\operatorname{cosec}\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\}[\sinh\left\{\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right\}\cos(BY+C)$$

$$-\ i\cosh\left\{\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right\}\sin(BY+C)]$$

3. Comparison of the situation reported in Section-2 with that was reported previously by Chakraborty and Chanda [7] for the Yang's R-gauge equations in real form

The Yang's R-gauge equations[5] read in real variables as stated below [5]

$$\phi(\phi_{11}+\phi_{22}+\phi_{33}+\phi_{44}) = (\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2)-(\psi_1^2+\psi_2^2+\psi_3^2+\psi_4^2)$$
$$-(\chi_1^2+\chi_2^2+\chi_3^2+\chi_4^2)-2(\psi_1\chi_2-\psi_2\chi_1+\psi_4\chi_3-\psi_3\chi_4) \quad (3.1a)$$

$$\phi(\psi_{11}+\psi_{22}+\psi_{33}+\psi_{44}) = 2(\phi_1\psi_1-\phi_2\psi_2+\phi_3\psi_3-\phi_4\psi_4)+2(\phi_1\chi_2-\phi_2\chi_1+\phi_4\chi_3-\phi_3\chi_4) \quad (3.1b)$$

$$\phi(\chi_{11}+\chi_{22}+\chi_{33}+\chi_{44}) = 2(\phi_1\chi_1-\phi_2\chi_2+\phi_3\chi_3-\phi_4\chi_4)+2(\phi_2\psi_1-\phi_1\psi_2+\phi_3\psi_4-\phi_4\psi_3) \quad (3.1c)$$

where $\rho=\psi+i\chi, \phi_1=\frac{\partial\phi}{\partial x^1}, \phi_{11}=\frac{\partial^2\phi}{\partial x^{1^2}}$

Chakraborty and Chanda [7] obtained the complex solutions with the attempt to generate exact solutions for the Yang's equations from some trivial solutions of the same equation. This is a part of the formalism due to Weiss, Tabor and Carnavale [18] in relation to Painleve properties for the partial differential equations.

However they reported the solutions to be

$$\phi=(2Hi/\zeta)(\ln\zeta) \quad (3.2a)$$

$$\psi=(H/\zeta)(\ln\zeta) \quad (3.2b)$$

$$\chi=(H/\zeta)(\ln\zeta) \quad (3.2c)$$

where H is an arbitrary constant and ζ satisfies

$$\zeta_{11}+\zeta_{22}+\zeta_{33}+\zeta_{44}=0 \quad (3.2d)$$

which are not correct. Solutions achievable through their procedure has been calculated by the present authors

$$\phi=\frac{i}{\sqrt{2}H\zeta\ln\zeta} \quad (3.3a)$$

$$\psi = \frac{1}{2H\zeta \ln \zeta} \tag{3.3b}$$

$$\chi = \frac{1}{2H\zeta \ln \zeta} \tag{3.3c}$$

where H is an arbitrary constant and ζ satisfies

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0$$

4. Comparison and summary

(i) Complex solutions for both the equations (2.2) and (3.1) are in terms of Laplace solutions. The solutions reported for Yang's equations expressed in terms of ζ which satisfied the Laplace equation in four dimension (Equation 3.2d). The solutions reported for the Generalized Charap's equations are expressed in terms of X and Y which are mutually conjugate Laplace solutions(Equation 2.3f, 2.3g, 2.3h, 2.3i).

(ii) In both the cases $\phi\phi^*$, $\psi\psi^*$, $\chi\chi^*$ can be expressed in the real form in a very straight forward way.

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