Annals of Pure and Applied Mathematics
Vol. 18, No. 1, 2018, 79-81
ISSN: 2279-087X (P), 2279-0888(online)
Published on 21 August 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v18n1a11

Annals of
Pure and Applied Mathematics

On Solutions to the Diophantine Equation $x^4 - y^2 = z^2$

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Received 18 July 2018; accepted 20 August 2018

Abstract. In this paper, we investigate solutions to the title equation. When y is prime, we determine the values x, y, z which form a solution to the equation, and show their connection to the Pythagorean triples $a^2 + b^2 = c^2$. We examine all triples in [7] up to $c = 2500$ ($c^4 = x^4 \leq (5 \text{ x}10)^8$), and establish that the equation has exactly three solutions. Two solutions in which x, y are primes, and one solution in which x is composite and y is prime. All the solutions are exhibited.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 2, 4]. The title equation stems from the equation $p^x + q^y = z^2$.

Whereas in most articles, the values x, y are investigated for the solutions of the equation, in this paper these values are fixed positive integers. In the equation $x^4 - y^2 = z^2$, x, y, z are positive integers and x, y are odd. In Section 2, the connection between the equation and Pythagorean triples is discussed. For all values $x \leq 2500$ and y prime, it is established that the equation has exactly three solutions.

2. Solutions to $x^4 - y^2 = z^2$ when y is prime

A set of positive integers a, b, c is called a "Pythagorean triple" (abbreviated triple) denoted (a, b, c) if $a^2 + b^2 = c^2$.

When y is prime, we determine the values x, y, z which form a solution to $x^4 - y^2 = z^2$, and establish the connection of these values to Pythagorean triples. This is done in Theorem 2.1.

Theorem 2.1. Let M be a positive integer. Suppose $\{(a, b, c)\}$ is the set of all Pythagorean triples where $(M^2 + (M + 1)^2)^{1/2} = c \leq 2500$. If y is prime, then there exist exactly three values M for which x, y, z

$$x = (M^2 + (M + 1)^2)^{1/2}, \qquad y = 2M + 1, \qquad z = 2M(M + 1) \qquad (1)$$

yield a solution of $x^4 - y^2 = z^2$.

Proof: The equation $x^4 - y^2 = z^2$ yields $y^2 = x^4 - z^2$ or $y^2 = (x^2 - z)(x^2 + z)$. When y is prime, all the possibilities for $x^2 - z$ are $x^2 - z = 1$, y, y^2. A priori, the values $x^2 - z = y$ and $x^2 - z = y^2$ are eliminated. Thus, $x^2 - z = 1$ and $x^2 + z = y^2$.

When $x^2 - z = 1$, then $x^2 = z + 1$ and $y^2 = 2z + 1$. Hence $y^2 - 1 = 2z$ or $2z = (y - 1)(y + 1)$. Observe that $2|(y - 1)$ and also $2|(y + 1)$. It is easily seen that it suffices to consider only one of these two cases. If $2|(y - 1)$, denote $2M = y - 1$ and $y = 2M + 1$. Thus, $2z = (2M)(2M + 2)$, and $z = 2M(M + 1)$. Since $x^2 + z = y^2$, it then follows that $x^2 = y^2 - z$ or $x^2 = (2M + 1)^2 - 2M(M + 1)$, and $x^2 = 2M^2 + 2M + 1 = M^2 + (M + 1)^2$ where M^2 and $(M + 1)^2$ are two consecutive squares and x^2 is odd. The values x, y, z in (1) have been determined, and only those values M which simultaneously yield a prime $y = 2M + 1$ and an integer x are considered.

We are therefore in search of particular values of M in triples which satisfy $a^2 = M^2$, $b^2 = (M + 1)^2$ and $x^2 = c^2 = M^2 + (M + 1)^2$. In [7] are listed "Pythagorean triples up to $c = 10000$". Examining all triples up to $c = 2500$, it turns out that only three triples satisfy our conditions, and yield solutions to $x^4 - y^2 = z^2$. These are:

Solution 1. $5^4 - 7^2 = 24^2$ $\quad x = 5, \quad y = 7, \quad z = 24, \quad M = 3.$

Solution 2. $29^4 - 41^2 = 840^2$ $\quad x = 29, \quad y = 41, \quad z = 840, \quad M = 20.$

In the above two solutions x and y are primes.

Solution 3. $169^4 - 239^2 = 28560^2$ $\quad x = 169, \quad y = 239, \quad z = 28560, \quad M = 119.$

In this solution x is composite (a square) and y is prime.

This completes the proof of Theorem 2.1. □

Remark 2.1. In view of $x^4 - y^2 = z^2$, a brief of results related to $p^4 + q^2 = z^2$ and to $p^3 \pm q^3 = z^2$ is as follows. In [2], $p^4 + q^2 = z^2$ is investigated. For $p = 2$ the equation has exactly one solution, whereas for each prime $p \geq 3$ two solutions exist. The connection to Pythagorean triples is also discussed. In [1], $p^3 \pm q^3 = z^2$ are considered when p and q are primes. For $q = 2$ both equations have no solutions, whereas for $3 \leq q < p \leq 101$ each equation has exactly one solution. It is presumed that when $p > 101$, more solutions may be found if a computer is used.

3. Conclusion

All triples $a^2 + b^2 = c^2$ where $c \leq 2500$ ($c^4 = x^4 \leq (5 \text{ x}10)^8$) have been examined. Besides **Solutions 1**, **2** and **3**, no other solutions have been found. As a consequence of the three solutions, the following questions may be raised.

Question 1. Do there exist other solutions of $x^4 - y^2 = z^2$ in which x, y are primes ?

Question 2. Do there exist other solutions of $x^4 - y^2 = z^2$ in which only one of x, y is prime ?

Question 3. Do there exist solutions of $x^4 - y^2 = z^2$ in which x, y are composites ?

Answers to these questions may be found in [7] for values $2500 < c \leq 10000$, and for $c > 10000$ by using a computer.

Conjecture. Suppose that x, y, z are positive integers. If y is prime, then the only solutions of the equation $x^4 - y^2 = z^2$ are

$$(x, y, z) = (5, 7, 24), \quad (29, 41, 840), \quad (169, 239, 28560).$$

If indeed the conjecture is true, then the solutions of $x^4 - y^2 = z^2$ are **Solutions 1**, **2** and **3**.

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