Annals of Pure and Applied Mathematics
Vol. 18, No. 1, 2018, 99-112
ISSN: 2279-087X (P), 2279-0888(online)
Published on 19 September 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v18n1a15

Annals of **Pure and Applied Mathematics**

Asymptotic Solutions of Coupled Spring Systems with Cubic Nonlinearity using Homotopy Perturbation Method

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Received 9 August 2018; accepted 15 September 2018

Abstract. In recent years, many analytical and numerical methods have emerged which are being used to obtain approximate solutions of a wide range of problems arising in mathematical modeling of linear and nonlinear physical and engineering problems. In this paper we study the application of the homotopy perturbation method (HPM) to obtain analytical approximate solutions of the nonlinear differential equations which model a coupled spring system with and without damping and external driving force. The application of the method is found to be justified by a good agreement between the results of HPM and the corresponding numerical ones obtained by using *Mathematica* 9.

Keywords: Homotopy perturbation method, Coupled spring equations, Damping, External force

AMS Mathematics Subject Classification (2010): 34E10, 37M99

1. Introduction

From the mathematical point of view, most of the real world physical and engineering problems are modeled as differential equations. While standard solution procedures exist for linear differential equations, nonlinear equations are rather difficult to solve, and in some cases, it is virtually impossible to find exact solutions. Mathematicians are in a constant search of new techniques to find analytically exact or approximate solutions for nonlinear ordinary and partial differential equations which model diverse fields of science and engineering. Some of the recently developed and popular methods used to find approximate solutions to nonlinear problems are the homotopy perturbation method (HPM) [1-7], the variational iteration method (VIM) [8-10], and the Adomian decomposition method [11, 12]. The homotopy perturbation method, introduced by the Chinese mathematician Dr. Ji Huan He in 1998, has come to be accepted as an elegant tool in the hands of researchers looking for simple yet highly effective solutions to complicated problems in many diverse areas of science and technology. It has been employed to solve a large variety of linear and nonlinear problems and found to provide highly accurate solutions in comparison with numerical techniques. In [13] and [14] He applied the HPM for solving nonlinear boundary value problem and Blasius differential equation, respectively. Ganji and Rafei used the HPM in [15] to obtain solitary wave solutions for a generalized nonlinear Hirota-Satsuma coupled KdV partial differential equations. This HPM has also been successfully applied to problems relating to the Laplace equation [16], heat radiation equations [17], nonlinear dispersive $K(mp)$ equations [18], nonlinear integral equations [19], nonlinear heat conduction and convection equations [20], nonlinear Schrödinger equations [21], nonlinear oscillators [22], nonlinear wave equations [23], nonlinear chemistry problems [24], and to other fields [25-32]. The HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions.

The aim of this article is to extend the application of the He's HPM to solve a system of nonlinear ordinary differential equations which give a mathematical model of coupled spring systems [33]. *Mathematica* 9.0 software has been used for computing and testing the accuracy of the analytical approximate HPM solutions compared with the numerical solutions.

2. Formulation of the Problem

The coupled spring system we study consists of two springs and two weights. One spring, having spring constant k_1 is attached to the ceiling and a weight of mass m_1 is attached to the lower end of this spring. To this weight, a second spring is attached having spring constant k_2. To the bottom of this second spring, a weight of mass m_2 is attached and the entire system appears as illustrated in **Figure 1**. Allowing the system to come to rest in equilibrium, we measure the displacement of the centre of mass of each weight from equilibrium as a function of time, and denote these measurements by $x_1(t)$ and $x_2(t)$ respectively.

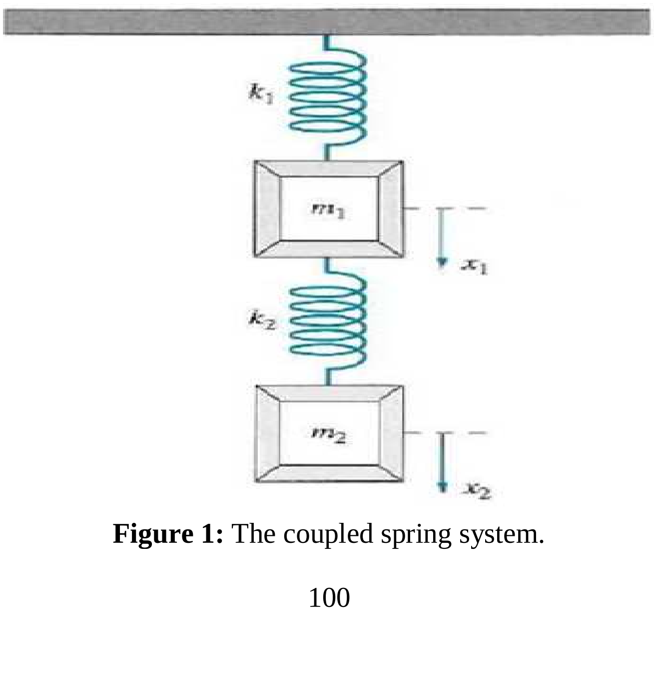

Figure 1: The coupled spring system.

Assuming Hooke's Law, under the assumption of small oscillations, the restoring forces are of the form $-k_1 l_1$ and $-k_2 l_2$ where l_1 and l_2 are the elongations (or compressions) of the two springs. Since the upper mass is attached to both springs, there are two restoring forces acting upon it: an upward restoring force $-k_1 x_1$ exerted by the elongation (or compression) x_1 of the first spring; an upward force $-k_2(x_2 - x_1)$ from the second spring's resistance to being elongated (or compressed) by the amount of $x_2 - x_1$. The second mass only 'feels' the restoring force from the elongation (or compression) of the second spring. If we assume that there are no damping forces present, then Newton's Law implies that the two equations representing the motions of the two weights are

$$m_1 x_1'' = -k_1 x_1 - k_2(x_1 - x_2) \tag{1}$$

$$m_2 x_2'' = -k_2(x_2 - x_1) \tag{2}$$

where primes denote derivation with respect to time. Thus we have a pair of coupled second-order linear differential equations.

The most common type of damping encountered in beginning courses is that of viscous damping; the damping force is proportional to the velocity. The damping of the first weight depends solely on its velocity and not the velocity of the second weight, and vice versa. We assume that the damping coefficients δ_1 and δ_2 are small. We add viscous damping to the model by adding the term $-\delta_1 x_1'$ to the equation (1) and $-\delta_2 x_2'$ to the equation (2).

If we assume that the restoring forces are nonlinear, which are most certainly the cases of large vibrations, we can modify the model accordingly. Rather than assuming that the restoring force is of the form $-kx$ (Hooke's law), we assume the restoring force has the form $-kx + \mu x^3$. We add nonlinearity to the model by adding the terms $\mu_1 x_1^3$ and $\mu_2(x_1 - x_2)^3$ to the equation (1) and $\mu_2(x_2 - x_1)^3$ to the equation (2). The range of motions for such nonlinear model is much more complicated than that for the corresponding linear model. An idea of this range of motions for a single spring model is given in [34]. Moreover, accuracy questions arise when solving these equations. No numerical solution can be expected to remain accurate over long time intervals. The accumulated local truncation error, algorithm error, round off error, propagation error, etc., eventually force the numerical solution to be inaccurate. This is discussed in some detail in the interesting papers by Knapp and Wagon [35], and by Fay and Joubert [36, 37].

It is a simple matter to add external forcing to the model. Indeed, we can drive each weight differently. Suppose we assume simple sinusoidal forcing of the form $F\cos\omega t$. Then the model becomes

$$m_1 x_1'' = -\delta_1 x_1' - k_1 x_1 + \mu_1 x_1^3 - k_2(x_1 - x_2) + \mu_2(x_1 - x_2)^3 + F_1 \cos\omega_1 t \tag{3}$$

$$m_2 x_2'' = -\delta_2 x_2' - k_2(x_2 - x_1) + \mu_2(x_2 - x_1)^3 + F_2 \cos\omega_2 t \tag{4}$$

The range of motions for nonlinear forced models is quite vast. We can expect to find bounded and unbounded solutions (nonlinear resonance), periodic solutions that share the period with the forcing (called harmonic solutions) and solutions that are periodic with a period of a multiple of the driving period (called sub harmonic solutions), and steady state periodic solutions (limit cycles in the phase plane). The conditions under which these motions occur are by no means easy to state.

3. Homotopy Perturbation Method

To illustrate the homotopy perturbation method, we consider a general equation of the type,

$$A(u(x)) - f(r) = 0,\ r \in \Omega \tag{5}$$

With the boundary conditions $B\left(u, \dfrac{du}{dx}\right) = 0,\ r \in \Gamma$ (6)

where A is the general differential operator, B is the boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytical function. Generally speaking, the operator A can be divided into a linear part *L* and a nonlinear part *N*. Now equation (5) can be written as:

$$L(u(x)) + N(u(x)) - f(r) = 0 \tag{7}$$

By the homotopy perturbation method, we construct a homotopy as $v(r, p) : \Omega \times [0,1] \to R$ which satisfies the following equation:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{8}$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of equation (5), which satisfies the boundary conditions. Considering equation (8), we will have

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{9}$$

and $H(v, 1) = A(v) - f(r) = 0$ (10)

The changing process of *p* from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy.

According to the homotopy perturbation theory, we can first use the embedding parameter *p* as a small parameter and assume that the solution of equation (8) can be written as a power series in *p* as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots\ldots\ldots\ldots\ldots \tag{11}$$

To obtain the approximate solution of equation (1) setting $p = 1$, we have

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots\ldots\ldots\ldots\ldots \tag{12}$$

The equation (12) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$.

4. Application of the Homotopy Perturbation Method (HPM)

Writing x for x_1 and y for x_2 we may rewrite the equations (3) and (4) as follow:

$$m_1 x'' = -\delta_1 x' - k_1 x - k_2(x-y) + \mu_1 x^3 + \mu_2 (x-y)^3 + F_1 \cos\omega_1 t \tag{13}$$

$$m_2 y'' = -\delta_2 y' - k_2(y-x) + \mu_2 (y-x)^3 + F_2 \cos\omega_2 t \tag{14}$$

According to the equation (8), we consider the following homotopy for the (13) and (14) are

$$m_1 x'' + \delta_1 x' + k_1 x - k_2(y-x) + p\left[-\mu_1 x^3 + \mu_2 (y-x)^3 - F_1 \cos\omega_1 t\right] = 0 \tag{15}$$

$$m_2 y'' + \delta_2 y' + k_2(y-x) + p\left[-\mu_2 (y-x)^3 - F_2 \cos\omega_2 t\right] = 0 \tag{16}$$

As outlined above, the basic assumption is that the solutions of equations (13) and (14) can be written as power series in *p*:

$$x = x_0 + p x_1 + p^2 x_2 + \ldots\ldots\ldots\ldots \tag{17}$$

$$y = y_0 + p y_1 + p^2 y_2 + \ldots\ldots\ldots\ldots \tag{18}$$

Therefore, substituting (17) and (18) into (15) and (16), and then equating the terms with identical powers of *p*, we can obtain the following set of linear differential equations:

$$p^0 : \begin{cases} k_1 x_0 + k_2 x_0 - k_2 y_0 + \delta_1 x_0' + m_1 x_0'' = 0 \\ -k_2 x_0 + k_2 y_0 + \delta_2 y_0' + m_2 y_0'' = 0 \end{cases}$$

$$p^1 : \begin{cases} -F_1 \cos t\omega_1 - \mu_1 x_0^3 - \mu_2 x_0^3 + k_1 x_1 + k_2 x_1 + 3\mu_2 x_0^2 y_0 - 3\mu_2 x_0 y_0^2 \\ +\mu_2 y_0^3 - k_2 y_1 + \delta_1 x_1' + m_1 x_1'' = 0 \qquad x_1(0) = 0,\ x_1'(0) = 0 \\ -F_2 \cos t\omega_2 + \mu_2 x_0^3 - k_2 x_1 - 3\mu_2 x_0^2 y_0 + 3\mu_2 x_0 y_0^2 - \mu_2 y_0^3 + k_2 y_1 \\ +\delta_2 y_1' + m_2 y_1'' = 0 \qquad y_1(0) = 0,\ y_1'(0) = 0 \end{cases}$$

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Case I (undamped undriven motion): In the absence of damping and external driving force, i.e., $F_1 = 0, F_2 = 0, \delta_1 = 0, \delta_2 = 0$, we choose $m_1 = m_2 = 1$, $k_1 = 0.4$, $k_2 = 1.808$, $\mu_1 = -0.16$, $\mu_2 = -0.1$. Then subject to the initial conditions $x(0) = 0.005$, $y(0) = 0.001$, the first few approximations of the homotopy perturbation solutions for equations (13) and (14) are derived in the following forms:

$$x_0 = 0.0027221 e^{2.602\times10^{-18} t} \cos 0.435t + 0.0022779 \cos 1.956t$$

$$y_0 = 0.00303982 e^{2.602\times10^{-18} t} \cos 0.435t - 0.00203982 \cos 1.956t$$

$$x_1 = (-2.95646\times10^{-7} e^{2.602\times10^{-18}t} + 2.93066\times10^{-7} e^{7.806\times10^{-18}t})\cos 0.435t$$
$$+(7.57125 - 7.43861 e^{5.204\times10^{-18}t})\times10^{-8}\cos 1.956t + \{2.38093\times10^{8}$$
$$(e^{2.602\times10^{-18}t} - \mathrm{e}^{7.806\times10^{-18}t}) - 1.78825\times10^{-9} e^{2.602\times10^{-18}t} t\}\sin 0.4347t$$
$$+\{1.13789\times10^{8}(1 - e^{5.204\times10^{-18}t}) - 1.82433\times10^{-9} t\}\sin 1.956t$$
$$y_1 = (-3.11678\times10^{-7} e^{2.602\times10^{-18}t} + 3.13196\times10^{-7} e^{7.806\times10^{-18}t})\cos 0.4347t$$
$$+1.38875\times10^{-9} e^{5.204\times10^{-18}t}\cos 1.08686t - 1.14846\times10^{-9} e^{2.602\times10^{-18}t}\cos 1.304t$$
$$-1.10911\times10^{-9}\cos 1.956t + \{2.65883\times10^{8}(e^{2.602\times10^{-18}t} - e^{7.806\times10^{-18}t})$$
$$-1.99697\times10^{-9} e^{2.602\times10^{-18}t} t\}\sin 0.435t - \{1.01896\times10^{8}(1 - e^{5.204\times10^{-18}t})$$
$$-1.63365\times10^{-9} t\}\sin 1.956t$$

Therefore, the solutions up to first approximations of the equations (13) and (14) are $x = x_0 + x_1$ and $y = y_0 + y_1$

Table 1: Comparison between the HMP and Numerical results for $x(t)$ and $y(\mathrm{t})$

Time (t)	**Homotopy Results**		**Numerical Results**		**Errors in %**	
	$x(t)$	$y(\mathrm{t})$	$x(t)$	$y(\mathrm{t})$	$x(t)$	$y(\mathrm{t})$
0	0.005	0.001	0.005	0.001	0	0
4	-0.00038953	-0.0005673	-0.00038956	-0.00056727	0.009851	0.00474139
8	-0.00484392	-0.00083333	-0.00484375	-0.00083347	0.0033871	0.0166006
12	0.00111795	0.00164481	0.0011179	0.00164477	0.0044629	0.00222489
16	0.00439269	0.00035231	0.00439243	0.00035257	0.0057609	0.0729195
20	-0.00169985	-0.00255738	-0.0016998	-0.00255735	0.0025688	0.00095026
24	-0.00369565	0.00038805	-0.00369523	0.00038772	0.0114081	0.0840284
28	0.00205594	0.00321559	0.00205596	0.00321561	0.0011179	0.00054631
32	0.00282861	-0.00130321	0.00282813	-0.00130275	0.0168171	0.0356236
36	-0.00213242	-0.00355852	-0.00213246	-0.00355848	0.0017270	0.00118794
40	-0.00188585	0.00228806	-0.00188525	0.00228751	0.0318059	0.0240745

This fact is graphically shown in **Figure 2.**

Case II (damped undriven motion): In the absence of the external forces, that is $F_1 = 0,\ F_2 = 0,$ we choose $m_1 = 1,\ m_2 = 1,\ k_1 = 3,\ k_2 = 2,\ \delta_1 = 0.1,\ \delta_2 = 0.2,\ \mu_1 = 0.16,$ $\mu_2 = 0.1$. Then subject to the initial conditions $x(0) = 0.6$, $y(0) = 0.2$, the first few approximations of the homotopy perturbation solutions for equations (13) and (14) are derived in the following forms:

$$x_0 = 0.199706 e^{-0.09t}\cos 0.996t + 0.400294 e^{-0.06t}\cos 2.448t$$
$$+0.005252 e^{-0.09t}\sin 0.996t + 0.015012 e^{-0.06t}\sin 2.448t$$
$$y_0 = 0.399891 e^{-0.09t}\cos 0.996t - 0.199891 e^{-0.06t}\cos 2.448t$$
$$+0.01848 e^{-0.09t}\sin 0.996t + 0.002288 e^{-0.06t}\sin 2.448t$$

$$x_1 = 0.000636e^{-0.24t}\cos 0.456t + (0.000051e^{-0.27t} + 0.000199e^{-0.21t} - 0.001828e^{-0.09t})\cos 0.996t + (0.000628e^{-0.24t} + 0.002815e^{-0.18t} - 0.002431e^{-0.06t})\cos 2.448t - 0.000051e^{-0.27t}\cos 2.988t + 0.000255e^{-0.21t}\cos 3.901t - 0.000268e^{-0.24t}\cos 4.441t + 0.000049e^{-0.21t}\cos 5.893 - 0.000177e^{-0.18t}\cos 7.345t - 0.000071e^{-0.24t}\sin 0.456t - (0.000869e^{-0.27t} + 0.01538e^{-0.21t} - 0.016485e^{-0.09t})\sin 0.996t - (0.008413e^{-0.24t} + 0.043656e^{-0.18t} - 0.052217e^{-0.06t})\sin 2.448 - 0.000071e^{-0.24t}\sin 4.441t$$

$$y_1 = -0.000268e^{-0.24t}\cos 0.456t + (0.000156e^{-0.27t} + 0.004426e^{-0.21t} - 0.00432e^{-0.09t})\cos 0.996t - (0.000154e^{-0.24t} - 0.000505e^{-0.18t})\cos 2.448t - 0.000051e^{-0.27t}\cos 2.988t - 0.000502e^{-0.21t}\cos 3.9t + 0.000084e^{-0.24t}\cos 4.44t - 0.00021e^{-0.21t}\cos 5.893t + 0.000178e^{-0.18t}\cos 7.345t - (0.001741e^{-0.27t} + 0.030824e^{-0.21t} - 0.032955e^{-0.09t})\sin 0.996t + (0.004201e^{-0.24t} + 0.021965e^{-0.18t} - 0.026086e^{-0.06t})\sin 2.448t$$

Therefore, the solutions up to first approximations of the equations (13) and (14) are $x = x_0 + x_1$ and $y = y_0 + y_1$

Table 2: Comparison between the HPM and Numerical results for $x(t)$ and y(t)

Time (t)	Homotopy Results		Numerical Results		Errors in %	
	$x(t)$	y(t)	$x(t)$	y(t)	$x(t)$	y(t)
0	0.6	0.2	0.6	0.2	0	0
4	-0.401964	-0.0522376	-0.401495	-0.0523665	0.116662	0.246265
8	0.199569	-0.100768	0.198919	-0.100408	0.327039	0.358525
12	-0.0598972	0.153063	-0.0595017	0.152605	0.664585	0.300552
16	-0.0090066	-0.10902	-0.00904281	-0.108719	0.400482	0.276927
20	0.0325736	0.028401	0.0323433	0.0283948	0.712161	0.0217638
24	-0.0387334	0.0340943	-0.0383576	0.0338563	0.979746	0.702848
28	0.0422335	-0.0572848	0.0417994	-0.0569481	1.03853	0.591232
32	-0.0449839	0.0492994	-0.0445549	0.0489943	0.962987	0.622844
36	0.043708	-0.0292358	0.0433348	-0.0290241	0.861144	0.729234
40	-0.0364871	0.0119506	-0.0362054	0.0118327	0.777912	0.99628

This fact is graphically shown in **Figure 3.**

Case III (undamped driven motion): In the absence of the damping, that is $\delta_1 = 0$, $\delta_2 = 0$, we choose $m_1 = 1$, $m_2 = 1$, $k_1 = 0.4$, $k_2 = 0.5$, $\mu_1 = -0.16$, $\mu_2 = -0.1$ $F_1 = 0.005$, $F_2 = 0.003$, $\omega_1 = 0.5$, $\omega_2 = 0.4$. Then subject to the initial conditions $x(0) = 0.005$,

$y(0) = 0.001$, the first few approximations of the homotopy perturbation solutions for equations (13) and (14) are derived in the following form:

$$x_0 = 0.00203576\cos 0.402t + 0.00296424\cos 1.113t$$

$$y_0 = 0.00300689\cos 0.402t - 0.00200689\cos 1.113t$$

$$\begin{aligned} x_1 = {} & (-7.48048\cos 0.309t - 3.1403\cos 0.402t - 4.04155\cos 1.113t \\ & + 1.12146\cos 1.206t - 2.75918t\sin 0.402t - 6.99461t\sin 1.113t)\times 10^{-9} \\ & + 0.9375\cos 0.4t - 0.0142857\cos 0.5t + (7.02903\cos 1.917t \\ & + 4.2854\cos 3.339t)\times 10^{-10} \end{aligned}$$

$$\begin{aligned} y_1 = {} & (-8.38175\cos 0.309t - 5.8168\cos 0.402t - 4.07541t\sin 0.402t \\ & + 4.73558t\sin 1.113t)\times 10^{-9} + 1.3875\cos 0.4t - 0.028571\cos 0.5t \\ & + 0.00147389\cos 1.113t + (6.9283\cos 1.824t - 5.47984\cos 1.206t \\ & - 2.13792\cos 1.917t + 3.25223\cos 2.628\,\mathrm{t} - 2.90135\cos 3.339t)\times 10^{-10} \end{aligned}$$

Therefore, the solutions up to first approximation of the equations (13) and (14) are

$$x = x_0 + x_1 \text{ and } y = y_0 + y_1$$

Table 3: Comparison between the HPM and Numerical results for $x(t)$ and $y(\mathrm{t})$

Time (t)	Homotopy Results		Numerical Results		Errors in %	
	$x(t)$	$y(\mathrm{t})$	$x(t)$	$y(\mathrm{t})$	$x(t)$	$y(\mathrm{t})$
0	0.005	0.001	0.005	0.001	0	0
4	0.0119992	0.0211882	0.011999	0.0211881	0.00113002	0.0005929
8	-0.0107062	-0.0122366	-0.0107067	-0.0122379	0.00441015	0.0103707
12	-0.0318465	-0.0551639	-0.0318378	-0.0551574	0.0271398	0.0117505
16	0.0244173	0.0390611	0.0244297	0.0390838	0.0506395	0.0579618
20	0.0420799	0.0696532	0.0420325	0.0696118	0.112767	0.0594703
24	-0.0382782	-0.0655714	-0.0383281	-0.0656552	0.13031	0.12754
28	-0.0437812	-0.0667641	-0.0436527	-0.0666414	0.294185	0.184104
32	0.0454436	0.0778186	0.04555	0.0779881	0.23342	0.217232
36	0.0437333	0.0601494	0.0434981	0.0599155	0.540698	0.390225
40	-0.0448386	-0.0734008	-0.0449987	-0.0736595	0.35578	0.35129

This fact is graphically shown in the **Figure 4.**

Case IV (damped driven motion): In the presence of both damping and external forces we choose $m_1 = 1$, $m_2 = 1$, $\delta_1 = 0.2$, $\delta_2 = 0.3$, $k_1 = 6$, $k_2 = 4$, $\mu_1 = 0.16$, $\mu_2 = 0.1$, $F_1 = 0.3$, $F_2 = 0.2$, $\omega_1 = 1$, $\omega_2 = 0.6$. Then subject to the initial conditions $x(0) = 0.7$, $y(0) = 0.2$, the first few approximations of the homotopy perturbation solutions for the equations (13) and (14) are derived in the following form:

$$\begin{aligned} x_0 = {} & 0.219825e^{-0.14t}\cos 1.407t + 0.480175e^{-0.11t}\cos 3.462t \\ & + 0.011219e^{-0.14t}\sin 1.407t + 0.019584e^{-0.11t}\sin 3.462t \end{aligned}$$

$$y_0 = 0.439932e^{-0.14t}\cos 1.407t - 0.239932e^{-0.11t}\cos 3.462t + 0.028659e^{-0.14t}\sin 1.407t - 0.001479e^{-0.11t}\sin 3.462t$$

$$x_1 = 0.041418\cos 0.6t + 0.000461e^{-0.39t}\cos 0.647t + 0.07768\cos t + (0.000237e^{-0.36t} - 0.1057e^{-0.14t})\cos 1.407t + (0.000453e^{-0.39t} + 0.0022885e^{-0.33t} - 0.01683e^{-0.11t})\cos 3.462t + 0.0001995e^{-0.36t}\cos 5.516t - 0.000193e^{-0.39t}\cos 6.277t + 0.000038e^{-0.36t}\cos 8.332t - 0.000153e^{-0.33t}\cos 10.386t + 0.004771\sin 0.6t - 0.000064e^{-0.39t}\sin 0.647t + 0.0149435\sin t - (0.000528e^{-0.42t} + 0.009359e^{-0.36t} + 0.0117e^{-0.14t})\sin 1.407t - (0.005563e^{-0.39t} + 0.029079e^{-0.33t} - 0.033743e^{-0.11t})\sin 3.462t - 0.000131\sin 5.924t - 0.000064e^{-0.39t}\sin 6.277t + 0.000131\sin 7.924t$$

$$y_1 = 0.099959\cos 0.6t - 0.000193e^{-0.39t}\cos 0.647t + 0.100541\cos t + (0.000069e^{-0.42t} + 0.003435e^{-0.36t} - 0.211357e^{-0.14t})\cos 1.407t - (0.000112e^{-0.39t} - 0.000116e^{-0.33t} - 0.007815e^{-0.11t})\cos 3.462t - 0.000069e^{-0.42t}\cos 4.222t - 0.000398e^{-0.36t}\cos 5.516t + 0.00006e^{-0.39t}\cos 6.277t - 0.000167e^{-0.36t}\cos 8.332t + 0.000153e^{-0.33t}\cos 10.386t + 0.010121\sin 0.6t + 0.030327\sin t - (0.001057e^{-0.42t} + 0.018746e^{-0.36t} + 0.026408e^{-0.14t})\sin 1.407t + (0.00278e^{-0.39t} + 0.014681e^{-0.33t} - 0.017128e^{-0.11t})\sin 3.462t$$

Therefore, the solutions up to first approximation of the equations (13) and (14) are

$x = x_0 + x_1$ and $y = y_0 + y_1$

Table 4: Comparison between HPM and Numerical results for $x(t)$ and y(t)

Time (t)	Homotopy Results		Numerical Results		Errors in %	
	$x(t)$	y(t)	$x(t)$	y(t)	$x(t)$	y(t)
0	0.7	0.2	0.7	0.2	0	0
4	0.0732052	-0.0988329	0.0715742	-0.0986097	2.27881	0.226382
8	-0.137313	0.110924	-0.136801	0.113806	0.37374	2.53262
12	-0.0247882	0.171342	-0.0228239	0.172811	8.6065	0.84995
16	-0.106899	-0.2396	-0.1062	-0.239424	0.658007	0.0735566
20	0.122812	0.108081	0.122675	0.108229	0.111443	0.136858
24	0.0192406	-0.0105786	0.0189694	-0.0106785	1.42985	0.935364
28	-0.112351	-0.134943	-0.112441	-0.135137	0.0802358	0.143812
32	0.103921	0.204301	0.103981	0.204273	0.0585195	0.0136497
36	-0.0571445	-0.132104	-0.0571877	-0.132158	0.0755187	0.0405423
40	-0.0210969	-0.0132315	-0.0211216	-0.0132385	0.116604	0.05299

This fact is graphically shown in **Figure 5.**

5. Result and discussion

The homotopy perturbation method is successfully applied to solve the nonlinear ordinary differential equations governing the motion of a coupled spring system. To test the accuracy of the HPM results, we match our results with the numerical results obtained by using *Mathematica* 9.0. The solution obtained by Homotopy perturbation method is an infinite series for appropriate initial conditions. Reasonably appropriate values for the spring constants k_1, k_2, damping constants δ_1, δ_2, nonlinear constants μ_1, μ_2, forcing constant F_1, F_2 and ω_1, ω_2 are taken for computing HPM solutions. Comparison between these and the corresponding numerical solutions computed by the *Mathematica* 9.0 program for various values of t are shown in the figures from **Figure 2** to **Figure 5** respectively in all four cases we study. These solutions represent a wide variety of interesting motions.

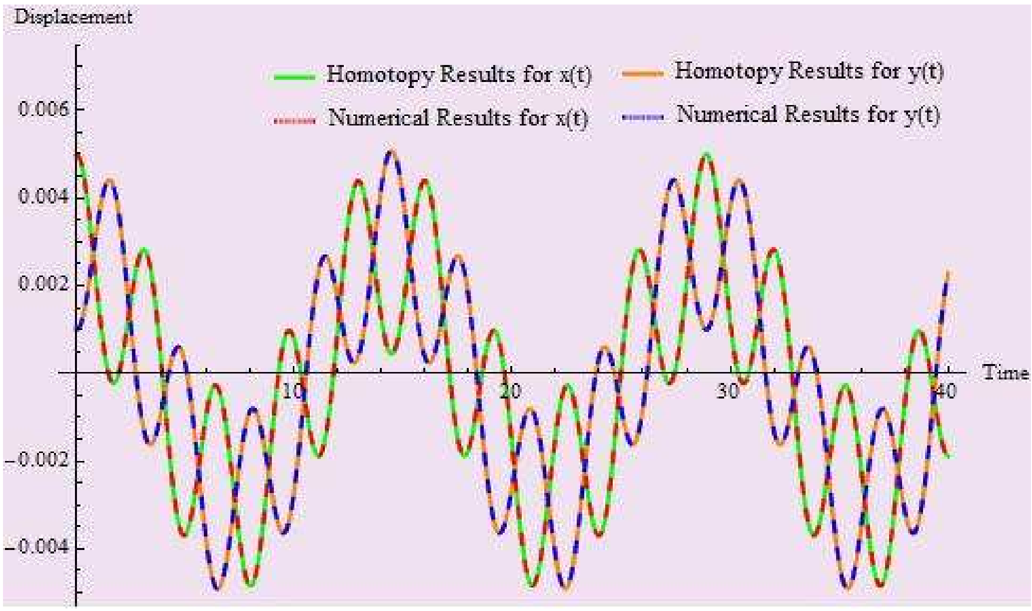

Figure 2: Comparison for $F_1 = 0$, $F_2 = 0$, $\delta_1 = 0$, $\delta_2 = 0$, $m_1 = 1$, $m_2 = 1$, $k_1 = 0.4$, $k_2 = 1.808$, $\mu_1 = -0.16$, $\mu_2 = -0.1$ and the initial conditions $x(0) = 0.005$, $y(0) = 0.001$.

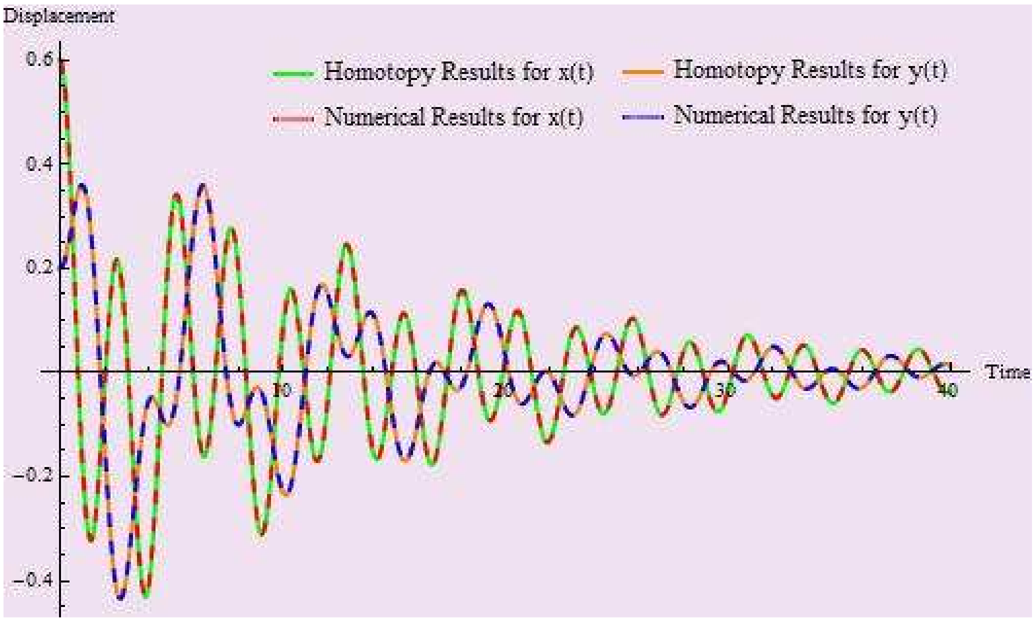

Figure 3: Comparison for $F_1 = 0$, $F_2 = 0$, $\delta_1 = 0.1$, $\delta_2 = 0.2$, $m_1 = 1$, $m_2 = 1$, $k_1 = 3$, $k_2 = 2$, $\mu_1 = 0.16$, $\mu_2 = 0.1$ and the initial conditions $x(0) = 0.6$, $y(0) = 0.2$.

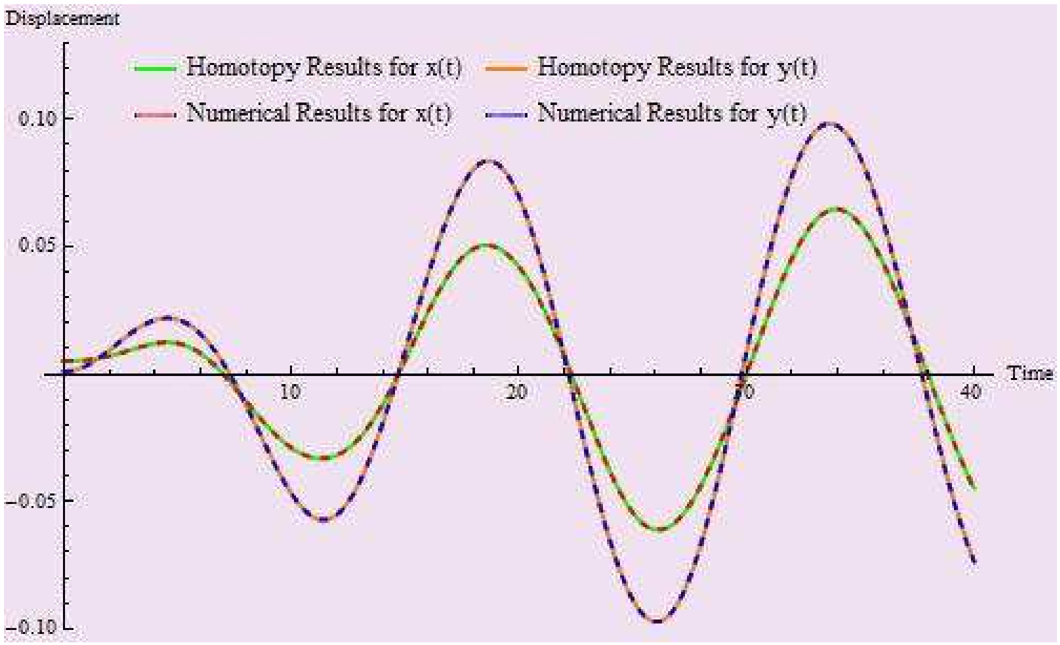

Figure 4: Comparison for $F_1 = 0.005$, $F_2 = 0.003$, $\delta_1 = 0$, $\delta_2 = 0$, $m_1 = 1$, $m_2 = 1$, $k_1 = 0.4$, $k_2 = 0.5$, $\mu_1 = -0.16$, $\mu_2 = -0.1$, $\omega_1 = 0.5$, $\omega_2 = 0.4$ and the initial conditions $x(0) = 0.005$, $y(0) = 0.001$.

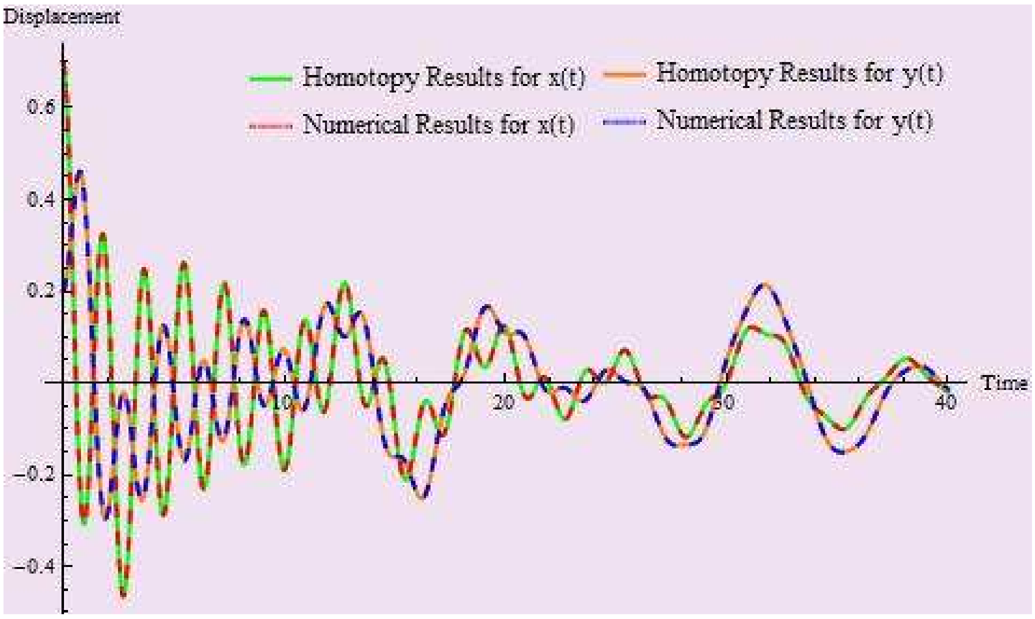

Figure 5: Comparison for $F_1 = 0.3$, $F_2 = 0.2$, $\delta_1 = 0.2$, $\delta_2 = 0.3$, $m_1 = 1$, $m_2 = 1$, $k_1 = 6$, $k_2 = 4$, $\mu_1 = 0.16$, $\mu_2 = 0.1$, $\omega_1 = 1$, $\omega_2 = 0.6$ and the initial conditions $x(0) = 0.7$, $y(0) = 0.2$.

6. Conclusion

In this article, the homotopy perturbation method (HPM) has been successfully applied to solve the dynamics of a coupled spring system with cubic nonlinearity. Our results suggest that it is an efficient method for obtaining solutions of nonlinear differential equations governing the motion of such problems. They also confirm the simplicity and efficiency of the method for solving any nonlinear ordinary differential equation or systems of nonlinear ordinary differential equations. It is also observed that the HPM is a promising method for solving other linear and nonlinear partial differential equations.

***Acknowledgements*:** This research is supported by Islamic University, Kushtia, Bangladesh. We like to thank all honourable members of the Academic Committee of the Department of Mathematics for providing us with logistic supports in carrying out the research. We also gratefully thank the anonymous reviewers for their valuable comments which helped us improve the manuscript significantly.

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