Annals of Pure and Applied Mathematics
Vol. 18, No. 1, 2018, 125-128
ISSN: 2279-087X (P), 2279-0888(online)
Published on 30 September 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v18n1a17

Annals of
Pure and Applied Mathematics

On the Non-Linear Diophantine Equation $p^x + (p+6)^y = z^2$

***Sani Gupta*[1], *Satish kumar*[2] and *Hari Kishan*[3]**

Department of Mathematics, D. N. College, Meerut, U.P., India
email: [2]skg22967@gmail.com, [3]harikishan10@rediffmail.com
[1]Corresponding author. email: sanigoel@gmail.com

Received 9 August 2018; accepted 29 September 2018

Abstract. In this paper, we consider the non-linear Diophantine equation $p^x + (p+6)^y = z^2$, where p and p+6 both are primes with p=6n+1. x, y and z are non-negative integers and n is natural number. It is shown that this non-linear Diophantine equation has no solution.

Keywords: Diophantine equations, exponential equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

A prime gap is the difference between two consecutive primes. There are infinitely many primes p for every positive integer k such that p+2k is also primes. If k= 3, the pairs (p, p+6) are known as sexy primes. The first ten such pairs are: (5,11), (7,13), (11,17), (13,19), (17,23), (23,29), (31,37), (37,43), (41,47), (47,53). Acu [1] proved that the Diophantine equation $2^x + 5^y = z^2$, where x, y and z are non-negative integers, has only two solutions (x, y, z) = (3, 0, 3) and (2, 1, 3). Sroysang [7] proved that the Diophantine equation $7^x + 8^y = z^2$ has the unique solution (x, y, z) = (0, 1, 3), where x, y and z are non-negative integers. Sroysang [9] proved that the Diophantine equation $5^x + 7^y = z^2$ has no solution, where x, y and z are non-negative integers. Sroysang [10] proved that the Diophantine equation $4^x + 10^y = z^2$ has no non-negative integer solution, where x, y and z are non-negative integers. Sroysang [11] proved that the Diophantine equation $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$ have no solution, where x, y and z are non-negative integers. Suvarnamani [8] discussed the Diophantine equation $p^x + (p+1)^y = z^2$ and found that (p, x, y, z) = (3, 1, 0, 2) is a unique solution, where p is an odd primes number and x, y and z are non-negative integers. Burshtein [3] obtained all the solutions of the Diophantine equation $p^x + (p+4)^y = z^2$ when p, (p+4) are primes and x + y = 2, 3, 4. Burshtein [4] discussed a Note on the Diophantine equation $2^a + 7^b = c^2$. The purpose of this Note is to complete the set of all solutions of this Diophantine equation. Burshtein [5] obtained all the solutions of the Diophantine equation $p^x + (p+6)^y = z^2$ when p, (p+6) are primes and x + y = 2, 3, 4. Burshtein [2] discussed on an open problem of Chatchaisthit on the Diophantine equation $2^x + p^y = z^2$ when p are particular primes and y=1. Kumar, Gupta and Hari Kishan [6] showed that the Diophantine equation $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ have no solution, where x, y and z are non negative integers.

On the Non-Linear Diophantine Equation $p^x + (p+6)^y = z^2$

In this paper, we consider the non-linear Diophantine equation

$$p^x + (p+6)^y = z^2 \qquad (1)$$

where p and p+6 both are sexy primes with p=6n+1 and x, y and z are non-negative integers and n is natural number. We use the congruency theory to solve the non-linear Diophantine equation. We discuss the solution of the Diophantine equations for many particular values of primes p like as 7, 13, 31, 37, 61, 67, 73, 97, 103, 151, 157, 193, 223, 271, 277, 307, 331, 367, 373, 433, 457, 541, 571, 601, 607, 613, 727, 733, 823, 853, 991.

2. Preliminaries

Lemma 2.1. If p = 6n+1 is primes, where n is natural number then the non-linear Diophantine equation $p^x + 1 = z^2$ has no solution, where x and z are non-negative integers.

Proof: Let x is non-negative integers then p^x is odd. Thus z^2 is even then $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$.

Since $p = 6n+1 \Rightarrow p \equiv 1 (mod\ 3) \Rightarrow p^x \equiv 1(mod\ 3) \Rightarrow p^x + 1 \equiv 2(mod\ 3)$.

Thus we obtain $z^2 \equiv 2(\text{mod}3)$. This is a contradiction.

Lemma 2.2. If p = 6n+1 is primes, where n is natural number then the Diophantine equation $1+ (p+6)^y = z^2$ has no solution, where y and z are non-negative integers.

Proof: Let y is non-negative integers then p^y is odd. Thus z^2 is even then $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$.

Since p = 6n+1 then $p \equiv 1 \pmod 3 \Rightarrow p \equiv 1 \pmod 3 \Rightarrow (p+6)^y \equiv 1(mod\ 3)$

$\Rightarrow 1+ (p+6)^y \equiv 2(mod\ 3)$.

Thus we obtain $z^2 \equiv 2(mod3)$. This is a contradiction.

3. Main theorem

Theorem 3.1. If p and p+6 both are primes with p=6n+1 then the non-linear Diophantine equation $p^x + (p+6)^y = z^2$ has no solution, where x, y, and z are non-negative integer and n is natural number.

Proof: Let x, y, and z be non-negative integers. Then there are three cases.

Case I. If x = 0, then by Lemma 2.2, the equation (1) has no solution.

Case II. If $x \geq 1$and y = 0, then by Lemma 2.1, the equation (1) also has no solution.

Case II. If $x \geq 1$and $y \geq 1$, then p^x and $(p+6)^y$ both are odd. Thus z^2 is even then $z^2 \equiv 0(\text{mod}3)$ or $z^2 \equiv 1(\text{mod}3)$.

Since p = 6n+1 then $p \equiv 1(mod3)$ and $(p+6) \equiv 1(mod3)$ so $p^x \equiv 1(mod3)$ and $(p+6)^y \equiv 1(mod3)$. Thus we obtain $p^x + (p+6)^y = z^2 \equiv 2(\text{mod}3)$, which is a contradiction. Hence the non-linear Diophantine equation (1) has no solution.

Let x, y, and z be non-negative integer and n is natural number. Then there are some corollaries given as below:

Corollary 3.1.1. If p and $p-6$ both are primes with p=6n+1 then the non-linear Diophantine equation $p^x + (p-6)^y = z^2$ has no solution.

Proof: If $(p+6) \equiv 1(mod3)$ then also $(p-6) \equiv 1(mod3)$ and $(p-6)^y \equiv 1(mod3)$.

Thus we obtain $p^x + (p-6)^y = z^2 \equiv 2(\text{mod}3)$, which is a contradiction.

Corollary 3.1.2. If p and $p-6$ both are primes with p=6n+1 then the non-linear Diophantine equation $(p-6)^x + p^y = z^2$ has no solution.
Proof: Replace p = p+6 in theorem 3.1.

Corollary 3.1.3. If p and $p+6$ both are primes with p=6n+1 then the non-linear Diophantine equation $(p+6)^x + p^y = z^2$ has no solution.
Proof: Replace p=p+6 in corollary 3.1.1.

Corollary 3.1.4. If p and $p+6$ both are primes with p=6n+1 then the non-linear Diophantine equation $p^x + (p+6)^y = w^{2k}$ has no solution.
Proof: Let $w^k = z$ then equation become $(p-6)^x + p^y = z^2$. By theorem 3.1 we implies that the equation has no solution.

Corollary 3.1.5. If p and $p+6$ both are primes with p=6n+1 then the non-linear Diophantine equation $p^x + (p+6)^y = w^{2k+4}$ has no solution.
Proof: Let $w^k = z$ then equation become $(p-6)^x + p^y = z^2$. By theorem 3.1 we implies that the equation has no solution.

Corollary 3.1.6. If p and $p+6$ both are primes with p=6n+1 then the non-linear Diophantine equation $p^{2n} + (p+6)^{2n} = w^{2n}$ has no solution.
Proof: Let $w^n = z$ then equation become $p^{2n} + (p+6)^{2n} = z^2$. Since p = 6n+1 then p $\equiv 1(mod 3)$ and (p+6) $\equiv 1(mod 3)$ so $p^{2n} \equiv 1(mod 3)$ and $(p+6)^{2n} \equiv 1(mod 3)$.
Thus we obtain $p^{2n} + (p+6)^{2n} = z^2 \equiv 2(\mathrm{mod}3)$, which is a contradiction.

Note 3.1.1. There are infinitely many primes p is of the form p=6n+1 which have no solution for any non negative integers x, y and z. Some non-linear Diophantine Equation for particular value of p between 1 to 100 are given in the table below.

$p^x + (p+6)^y = z^2$	Solution of the equation
$7^x + 13^y = z^2$	No Solution
$13^x + 19^y = z^2$	No Solution
$31^x + 37^y = z^2$	No Solution
$37^x + 43^y = z^2$	No Solution
$61^x + 67^y = z^2$	No Solution
$67^x + 73^y = z^2$	No Solution
$73^x + 79^y = z^2$	No Solution

4. Conclusion

In this paper, we find that for both primes p, p+6 and non negative integers x, y and z, the non-linear Diophantine equation $p^x + (p+6)^y = z^2$ has no solution if p of the form p = 6n+1, where n is natural number. There are infinitely many particular value of primes p for which this Diophantine equation has no solution. Some value of primes p between 1 to 1000 are 7, 13, 31, 37, 61, 67, 73, 97, 103, 151, 157, 193, 223, 271, 277, 307, 331, 367, 373, 433, 457, 541, 571, 601, 607, 613, 727, 733, 823, 853, 991.

REFERENCES

1. D.Acu, On the Diophantine equation $2^x + 5^y = z^2$, *Gen. Math.*, 15 (4) (2007)145-148.
2. N.Burshtein, On the open problem of S. Chatchaisthit on the diophantine equation $2^x + p^y = z^2$, *Annals of Pure and Applied Mathematics,* 16 (1) (2018) 31-35.
3. N.Burshtein, On the diophantine equation $p^x + (p+4)^y = z^2$, *Annals of Pure and Applied Mathematics*, 16 (1) (2018) 241-244.
4. N.Burshtein, On a the note on the diophantine equation $2^a + 7^b = c^2$, *Annals of Pure and Applied Mathematics*, 16 (2) (2018) 305-306.
5. N.Burshtein, On the diophantine equation $p^x + (p+6)^y = z^2$, *Annals of Pure and Applied Mathematics*, 17 (1) (2018) 101-106.
6. S.Kumar, S.Gupta and H.Kishan, On the non-linear diophantine equation $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$, *Annals of Pure and Applied Mathematics,* 18(1) (2018) 91-94.
7. B.Sroysang, On the diophantine equation $7^x + 8^y = z^2$, *International Journal Pure and Applied Mathematics,* 84 (2013) 111-114.
8. A.Suvarnamani, On the diophantine equation $p^x + (p+1)^y = z^2$, *International Journal Pure and Applied Mathematics,* 94 (5) (2014) 689-692.
9. B.Sroysang, On the diophantine equation $5^x + 7^y = z^2$, *International Journal Pure and Applied Mathematics,* 89 (1) (2014) 115-118.
10. B.Sroysang, On the diophantine equation $4^x + 10^y = z^2$, *International Journal Pure and Applied Mathematics*, 91 (1) (2014) 113-116.
11. B.Sroysang, On the diophantine equation $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$, *International Journal Pure and Applied Mathematics*, 92 (1) (2014) 113-116.