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Case Study of Non Strong Arc in Cartesian Product of Fuzzy Graphs

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Abstract. In this paper, we discussed the Cartesian product of the fuzzy graphs like fuzzy path and fuzzy cycle. We done case study of non strong arc in Cartesian product of fuzzy graph and deduced the formula for finding number of non strong arc in Cartesian product of the fuzzy graph.

Keywords: Fuzzy graph, Cartesian product, Strong arc, Non strong arc, Strong arc domination.

AMS Mathematics Subject Classification (2010): 05C72

1. Introduction

In1965, Zadeh introduced the concept of fuzzy subset of a set as a way for representing uncertainty [15]. Zadeh's ideas stirred the interest of researchers worldwide. Moderson and Nair discussed fuzzy graph theory fuzzy hypergraph [4]. Fuzzy graph is the generalization of the ordinary graph. The formal mathematical definition of domination was given by Ore in 1962 [10]. In 1975, Rosenfeld introduced the notion of fuzzy graph and several analogs of theoretic concepts such as path, cycle and connectedness [11]. Somasundaram and Somasundaram discussed the domination in fuzzy graph using effective arc[12]. Nagoorgani and Chandrasekarn discussed the strong arc in fuzzy graph [8,9]. Bhutani and Rosenfeld have introduced the concept of strong arcs in fuzzy graph [1,2]. Several works on fuzzy graph are also done by Pal and Rashmanlou [5,6], Methew and Sunitha [7], Samanta and Pal [14-20]. Before discuss case study of non strong arc in Cartesian product of fuzzy graph, we are placed few preliminary

2. Preliminaries [8, 13]

Definition 2.1. Fuzzy graph $G(\sigma,\mu)$ is pair of function, $V \rightarrow [0,1]$ and $\mu: V \times V \rightarrow [0,1]$ where for all $u, v\ in\ V$, we have $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$.

Definition 2.2. The fuzzy graph $H(\tau,\rho)$ is called a fuzzy subgraph of $G(\sigma,\mu)$ if $\tau(u) \leq \sigma(u)$ for all u in V and $\rho(u,v) \leq \mu(u,v)$ for all $u, v\ in\ V$.

Definition 2.3. A fuzzy subgraph $H(\tau,\rho)$ is said to be a spanning sub graph of G(σ,μ) if $\tau(u) = \sigma(u)$ for all u in V. In this case the two graphs have the same fuzzy node set, they differ only in the arc weights.

Definition 2.4. Let G(σ,μ) be a fuzzy graph and τ be fuzzy subset of σ, that is, $\tau(u) \leq \sigma(u)$ for all u in V. Then the fuzzy subgrpah of G(σ,μ) induced by τ is the maximal fuzzy subgraph of G(σ,μ) that has fuzzy node set τ. Evidently, this is just the fuzzy graph $H(\tau,\rho)$ where $\rho(u,v) = \tau(u) \wedge \tau(v) \wedge \mu(u,v)$ for all u,v in V.

Definition 2.5. The underlying crisp graph of a fuzzy graph G(σ,μ) is denoted by $G^* = (\sigma^*,\mu^*)$, where

$$\sigma^* = \{ u \in V \mid \sigma(u) > 0 \} \text{ and } \mu^* = \{ (u,v) \in V \times V \mid \mu(u,v) > 0 \}.$$

Definition 2.6. A fuzzy graph G(σ,μ) is a strong fuzzy graph if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u,v \in \mu^*$ and is a complete fuzzy graph if

$$\mu(u,v) = \sigma(u) \wedge \sigma(v) \text{ for all } u,v \text{ in } \sigma^*.$$

Two nodes u and v are said to be neighbors if $\mu(u,v) > 0$.

Definition 2.7. A fuzzy graph G = (σ, μ) is said to be Bipartite if the node set V can be Partitioned into two non empty sets V_1 and V_2 such that $\mu(v1, v2) = 0$ if $v_1, v_2 \epsilon V_1$ or $v_1, v_2 \epsilon V_2$. Further if $\mu(v_1, v_2) > 0$ for all $v_1 \epsilon V_1$ and $v_2 \epsilon V_2$ then G is called complete bipartite graph and it is denoted by $K_{\sigma 1,\sigma 2}$ where σ1 & σ2 are respectively the restriction of σ to V_1 and V_2

Definition 2.8. The complement of a fuzzy graph G(σ,μ) is a subgraph $\bar{G} = (\bar{\sigma},\bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\bar{\mu}(u,v) = \sigma(u) \wedge \sigma(v) - \mu(u,v)$ for all u,v in V.
A fuzzy graph is self complementary if $G = \bar{G}$

Definition 2.9. The order p and size q of a fuzzy graph G(σ,μ) is defined as $p = \sum_{u \in V} \sigma(u)$ and $q = \sum_{(u,v) \in E} \mu(u,v)$.

Definition 2.10. The degree of the vertex u is defined as the sum of weight of arc incident at u, and is denoted by d(u).

Definition 2.11. A Path ρ of a fuzzy graph G(σ,μ) is a sequence of distinct nodes v_1, v_2, $v_3, \ldots v_n$ such that $\mu(v_{i-1}, v_i) > 0$ where $1 \leq i \leq n$. A path is called a cycle if $u_0 = u_n$ and $n \geq 3$.

Definition 2.12. Let u, v be two nodes in G(σ,μ). If they are connected by means of a path ρ then strength of that path is $\wedge_{i=i}^{n} \mu(u_{i-1}, v_i)$.

Definition 2.13. Two nodes that are joined by a path are said to be connected. The relation connected is reflexive, symmetric and transitive. If u and v are connected by means of length k, then $\mu^k(u,v) = \sup \{\mu(u,v_1) \wedge \mu(v_1,v_2) \ldots \wedge \mu(v_{k-1},v_k) \mid u, v_1, v_2, \ldots v$ in such path ρ }

Definition 2.14. A Strongest path joining any two nodes u,v is a path corresponding to maximum strength between u and v. The strength of the strongest path is denoted by $\mu^{\infty}(u,v)$. $\mu^{\infty}(u,v) = \sup\{\mu^{k}(u,v) \mid k = 1,2,3\ldots\}$

Example 2.1.

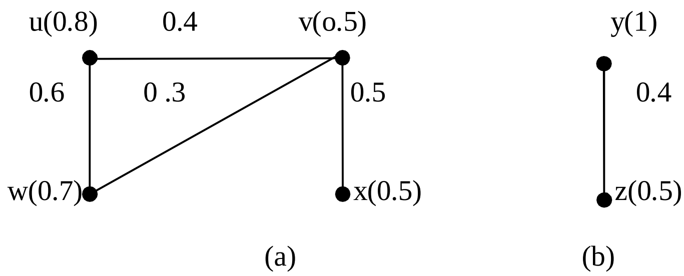

Figure 1:

In this fuzzy graph, Fig 1(a), u = w, v, x is a w-x path of length 2 and strength is 0.3.
Another path of w-x is w, u, v, x of length 3 and strength is 0.4.
But strength of the strongest path joining w and x is $\mu^{\infty}(w,x) = \sup\{0.3, 0.4\} = 0.4$

Definition 2.15. Let G(σ, μ) be fuzzy graph. Let x, y be two distinct nodes and G' be the fuzzy subgraph obtained by deleting the arc (x,y) that is $G'(\sigma, \mu')$ where $\mu'(x,y) = 0$ *and* $\mu' = \mu$ for all other. Then (x, y) is said to be fuzzy bridge in G if $\mu'^{\infty}(u,v) < \mu^{\infty}(u,v)$ for some u, v *in* V

Definition 2.16. A node is a fuzzy cut node of G(σ,μ) if removal of it reduces the strength of the connectedness between some other pair of nodes. That is, w is a fuzzy cut node of G(σ,μ) iff there exist u,v such that w is on every strongest path from u to v .

Definition 2.17. An arc (u, v) of the fuzzy graph G(σ,μ) is called an effective edge if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ and effective edge neighborhood of u $\in V$ is $N_e(u) = \{v \in V: edge(u,v)\ is\ effective\}$. $N_e[u] = N_e(u) \cup \{u\}$ is the closed neighbourhood of u.
The minimum cardinality of effective neighborhood $\delta_e(G) = \min\{|N_e(u)| \ u \in V(G)\}$.
Maximum cardinality of effective neighborhood $\Delta_e(G) = \max\{|N_e(u)| \ u \in V(G)\}$.

Domination in fuzzy graphs using strong arcs

Definition 2.18. An arc (u,v) of the fuzzy graph G(σ,μ) is called a strong arc if $\mu(u,v) = \mu^{\infty}(u,v)$ else arc(u,v) is called non strong. Strong neighborhood of u $\in V$ is $N_s(u) = \{v \in V: arc(u,v)$ is strong$\}$. $N_s[u] = N_s(u) \cup \{u\}$ is the closed neighborhood of u. The minimum cardinality of strong neighborhood $\delta_s(G) = \min\{|N_s(u)|: u \in V(G)\}$. Maximum cardinality of strong neighborhood $\Delta_s(G) = \max\{|N_s(u)|: u \in V(G)\}$.

Definition 2.19. Let G(σ,μ) be a fuzzy graph. Let u, v be two nodes of G(σ,μ). We say that u dominates v if edge (u,v) is a strong arc. A subset D of V is called a dominating set of G(σ,μ) if for every $v \in V - D$, there exists $u \in D$ such that u dominates v. A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set. The minimum fuzzy cardinality taken over all dominating sets of a graph G is called the strong arc domination number and is denoted by $\gamma_s(G)$ and the corresponding dominating set is called minimum strong arc dominating set.. The number of elements in the minimum strong arc dominating set is denoted by n[$\gamma_s(G)$]

Example 2.20. In fig(i)(a), (u, v), (u, w), (v, x) are strong arcs and (v, w) is non strong arc. D_1 = {u, x} ,D_2 = {w, x}, D_3 = {w, v} ,D_4 = {u,v} are dominating sets. Also D_1, D_2, D_3, D_4 are minimal dominating sets. Therefore $|D_1| = 0.8 + 0.5 = 1.3, |D_2| = 0.7 + 0.5 = 1.2, |D_3| = 0.5 + 0.7 = 1.2$ *and*
$|D_4| = 0.8 + 0.5 = 1.3$. Therefore, min{1.3, 1.2, 1.2, 1.3 } = 1.2.
Hence D_1 and D_2 are minimum dominating sets. $\gamma_s = 1.2$ and n[γ_s [G] = 2.

3. Case study of non strong arc in Cartesian product of fuzzy graph

Consider the Cartesian product G(V,X) = G_1 X G_2 of G_1 and G_2. Then V= V_1xV_2 and X= { ((u,u_2),(u,v_2)) / u ∈ V_1, (u_2 ,v_2) ∈ X_2 } U {((u_1,w),(v_1,w)) / w ∈ V_2,(u_2,v_1) ∈ X_1 } .

Definition 3.1. Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of X_i i = 1,2. Define the fuzzy subsets $\sigma_1 \; X \; \sigma_2$ of V and $\mu_1 \; X \; \mu_2$ of X as follows:

$(\sigma_1 \; X \; \sigma_2 \,)(\, u_1, u_2) = \min\{ \sigma_1 \; (u_1), \sigma_2(u_2)\} \; \forall (u_1, u_2) \in V$

$(\mu_1 \; X \; \mu_2)\big((u, u_2), (u, v_2)\big) = \; \min \{ \, \sigma_1 \; (u_1), \mu_2(u_2, v_2)\} \; \forall \; u \; \in \; V_1 \forall \; (u_2, v_2) \in \; X_2 \; and$

$(\mu_1 \; X \; \mu_2)\big((u_1, w), (v_1, w)\big) = \; \min \{ \, \sigma_2 \; (w), \mu_1(u_1, v_1)\} \; \forall \; w \; \in V_2 \; and \; \forall \; (u_1, v_1) \in X_1$

Then the fuzzy graph G($\sigma_1 \; X \; \sigma_2, \mu_1 \; X \; \mu_2$) is said to be the Cartesian product of $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$.

Theorem 3.2. If any two vertices of fuzzy graph G(σ, μ) are connected by exactly one path then every arc of G(σ, μ) are strong.

Proof: Let G(σ, μ) be a connected fuzzy graph and let n be the number vertices of G. Take n = 2 , there must be u and v adjoined by one arc (Since G is connected fuzzy graph) .Clearly , $\mu^\infty(u.v) = \sup\{\, \mu(u,v)\} = \mu(u,v)$. Therefore , arc (u, v) is strong. Assume that n > 2. In a fuzzy path, the $\mu^\infty(u.v)$ of any arc in the path will be same fuzzy value $\mu(u,v)$ of the arc (u,v) since connected by one path. By the above argument ,evidently it is proved that $\mu^\infty(u.v) = \; \mu(u,v)$ for any number of arc in a given path .Hence all the arc are strong.

Theorems 3.3. Let G(σ, μ) be a fuzzy cycle in which lowest fuzzy value of an arc occurs at more than once ,then that arc must be strong.

Proof: Let G(σ, μ) be a fuzzy cycle as in figure (ii) and let $\mu(u,v) \leq \mu(v,w) \leq \mu(u,w)$

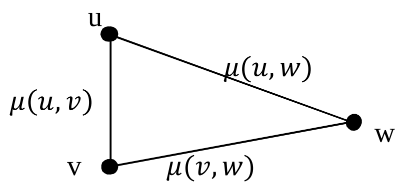

Figure 2:

Suppose , $\mu(u,v) < \mu(v,w) \leq \mu(u,w)$, then obviously arc (u, v) is non strong .

If not $\mu(u,v) = \mu(v,w)$,then $\mu^{\infty}(u.v) = \sup\{\mu(u,v), \mu(v,w)\} = \mu(u,v)$
(Since there are two paths connecting u and v and $\mu(u,v) = \mu(v,w)$)
Hence, arc (u, v) is strong (by definition 3.1)

Example 3.1.

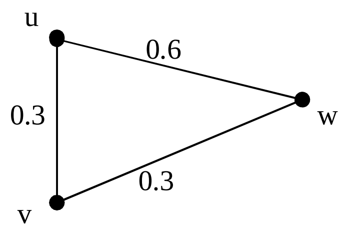

Figure 3:

$\mu(u,v) = \mu(v,w) = 0{,}3$, then $\mu^{\infty}(u.v) = \sup\{\mu(u,v), \mu(v,w)\}$
$\mu^{\infty}(u.v) = \sup\{0.3, 0.3\} = 0.3 = \mu(u,v)$. Hence, arc (u,v) is strong .
Note that **,** In the above example, all arc are strong.

Theorem 3.4. Let G($\sigma_1\ X\ \sigma_2,\ \mu_1\ X\ \mu_2$) be the Cartesian product of fuzzy graph $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$. If $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ doesn't have any non strong arc then G($\sigma_1\ X\ \sigma_2, \mu_1\ X\ \mu_2$) doesn't have any non strong arc.

In other words, if all the arcs of G_1 and G_2 are strong then all the arcs of G_1 X G_2 are strong.

Proof: Assume that $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ are as in Fig. 4 (a) ,(b)

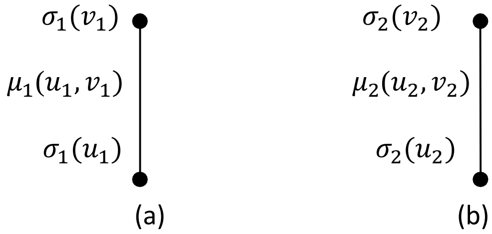

Figure 4:

The Cartesian product of fuzzy graph G_1 x G_2 of G_1 and G_2 are drawn as in fig (iv)

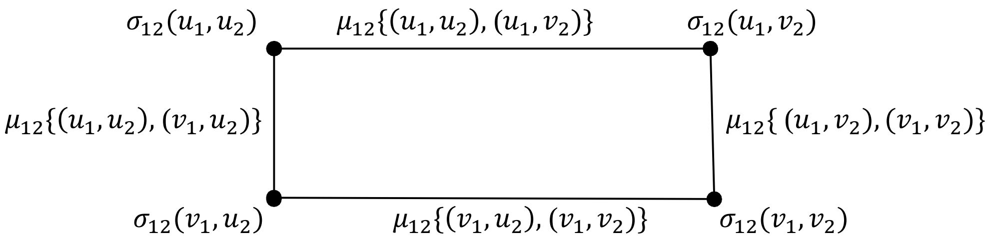

Figure 5:

Here G_1 and G_2 are fuzzy path (or any fuzzy graph).
Clearly all the arcs of G_1 and G_2 are strong arc.(by theorem 3.1)

We have to prove that all the arcs of G_1 X G_2 are strong. It is enough that we prove all the four arcs of fig (iv) are strong arc. The proof of the above hypothesis is discussed in the three four cases below.

Case 1:

Suppose $\sigma_1(u_1) < \mu_2(u_2, v_2)$ *and* $\sigma_1(v_1) < \mu_2(u_2, v_2)$ (1)

By definition 2.1, $\mu_1(u_1, v_1) \le \sigma_1(v_1) \wedge \sigma_1(u_1)$ and $\mu_2(u_2, v_2) \le \sigma_2(u_2) \wedge \sigma_2(v_2)$

Now, $\sigma_1(v_1) < \mu_2(u_2, v_2) \le \sigma_2(u_2) \wedge \sigma_2(v_2)$ (by (1)

$$\left.\begin{aligned} &\text{Hence, we must have } \sigma_1(u_1) < \sigma_2(u_2) \text{ and } \sigma_1(u_1) < \sigma_2(v_2) \\ &\text{Similarly, } \sigma_1(v_1) < \sigma_2(u_2) \text{ and } \sigma_1(v_1) < \sigma_{s2}(v_2) \end{aligned}\right\} \quad (2)$$

By definition of Cartesian product, a vertex u_1 of $\sigma_1(u_1)$ of G_1 can contribute two new vertices of $\sigma_{12}(u_1, u_2)$, $\sigma_{12}(u_1, v_2)$ and $\mu_{12}\{(u_1, u_2), (u_1, v_2)\}$ in G_1 X G_2. Similarly, v_1 can be done in the same method. As in fig (iv), following are the four arcs, whose fuzzy values are to be found.

Now, $\mu_1 \mathrm{X} \mu_2\{(u_1, u_2), (u_1, v_2)\}$ = min{ $\sigma_1(u_1), \mu_2(u_2, v_2)\}$ = $\sigma_1(u_1)$ (by (1))

$\mu_1 \mathrm{X} \mu_2$ $\{(v_1, u_2), (v_1, v_2)\}$ = min { $\sigma_1(v_1), \mu_2(u_2, v_2)\}$ = $\sigma_1(v_1)$ (by (1))

$\mu_1 \mathrm{X} \mu_2$ $\{(u_1, u_2), (v_1, u_2)\}$ = min { $\mu_1(u_1, v_1)$, $\sigma_2(u_2)\}$ = $\mu_1(u_1, v_1)$

$\mu_1 \mathrm{X} \mu_2$ $\{(u_1, v_2), (v_1, v_2)\}$ = min { $\mu_1(u_1, v_1)$, $\sigma_2(v_2)\}$ = $\mu_1(u_1, v_1)$

{ For, $\mu_1(u_1, v_1) \le \sigma_1(u_1) \wedge \sigma_1(v_1) < \sigma_2(u_2) \wedge \sigma_2(v_2)$ by (2)

Hence, $\mu_1(u_1, v_1) < \sigma_2(u_2)$ and $\mu_1(u_1, v_1) < \sigma_2(v_2)\}$

Now, the above arcs value are to be plotted in fig (iv). The fuzzy graph G_1 X G_2 becomes

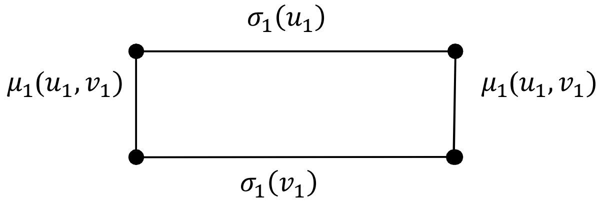

Figure 6:

Since, $\mu_1(u_1, v_1) \le \sigma_1(u_1) \wedge \sigma_1(v_1)$ and (by theorem 3.2, Let $G(\sigma, \mu)$ be a fuzzy cycle in which lowest fuzzy value of an arc occurs at more than once ,then it must be strong arc.) . Hence, all the arcs are strong in G_1 X G_2.

Case 2:

Suppose $\sigma_1(u_1) > \mu_2(u_2, v_2)$ and $\sigma_1(v_1) > \mu_2(u_2, v_2)$ (3)

$\mu_1 \mathrm{X} \mu_2\{(u_1, u_2), (u_1, v_2)\}$ = min{ $\sigma_1(u_1), \mu_2(u_2, v_2)\}$ = $\mu_2(u_2, v_2)$ (by (3))

$\mu_1 \mathrm{X} \mu_2$ $\{(v_1, u_2), (v_1, v_2)\}$ = min { $\sigma_1(v_1), \mu_2(u_2, v_2)\}$ = $\mu_2(u_2, v_2)$ (by (3))

$$\left.\begin{aligned} &\mu_1 \mathrm{X} \mu_2 \{(u_1, u_2), (v_1, u_2)\} = \min\{\mu_1(u_1, v_1), \sigma_2(u_2)\} \\ &\mu_1 \mathrm{X} \mu_2 \{(u_1, v_2), (v_1, v_2)\} = \min\{\mu_1(u_1, v_1), \sigma_2(v_2)\} \end{aligned}\right\} = \text{any fuzzy value but} \ge$$

$\mu_2(u_2, v_2)$ (or) $\ge$ $\mu_1(u_1, v_1)$.

For,

If $\mu_1(u_1, v_1) \ge \mu_2(u_2, v_2)$ then we must have $\sigma_2(u_2) \ge \mu_2(u_2, v_2)$, $\sigma_2(v_2) \ge \mu_2(u_2, v_2)$ and if $\mu_1(u_1, v_1) \le \mu_2(u_2, v_2)$ then we must have $\sigma_2(u_2) \ge \mu_1(u_1, v_1)$, $\sigma_2(v_2) \ge \mu_1(u_1, v_1)$

In this case $\mu_2(u_2, v_2)$ *or* $\mu_1(u_1, v_1)$ is the lowest fuzzy value of other arcs of G_1 X G_2 and it can occur more than once. By theorem 3.2, all the arcs of G_1 X G_2 are strong.

Case 3:

Suppose, $\sigma_1(u_1) < \mu_2(u_2, v_2)$ and $\sigma_1(v_1) > \mu_2(u_2, v_2)$ (4)

$\mu_1 \mathrm{X} \mu_2\{(u_1, u_2), (u_1, v_2)\}$ = min{ $\sigma_1(u_1), \mu_2(u_2, v_2)\}$ = $\sigma_1(u_1)$ (by (4))

$\mu_1 \mathrm{X} \mu_2$ $\{(v_1, u_2), (v_1, v_2)\}$ = min { $\sigma_1(v_1), \mu_2(u_2, v_2)\}$ = $\mu_2(u_2, v_2)$ (by (4))

$\mu_1 \mathrm{X} \mu_2$ $\{(u_1, u_2), (v_1, u_2)\}$ = min { $\mu_1(u_1, v_1)$, $\sigma_2(u_2)\}$ = $\mu_1(u_1, v_1)$

$\mu_1 \mathrm{X} \mu_2$ $\{(u_1, v_2), (v_1, v_2)\}$ = min { $\mu_1(u_1, v_1)$, $\sigma_2(v_2)\}$ = $\mu_1(u_1, v_1)$

For,

Now, $\mu_1(u_1, v_1) \leq \sigma_1(u_1) < \mu_2(u_2, v_2)$.

Therefore, we must have $\mu_1(u_1, v_1) < \sigma_2(u_2)$ and $\mu_1(u_1, v_1) < \sigma_2(v_2)$. From (4), we also have $\sigma_2(u_2) > \sigma_1(u_1)$, and $\sigma_2(v_2) > \sigma_1(u_1)$.

In this case, $\mu_1(u_1, v_1)$ is the lowest value of all other arcs of G_1 X G_2 and this arc occurs more than once in fuzzy cycle. Hence all the arcs are strong.

Case 4:

Suppose $\sigma_1(u_1) = \mu_2(u_2, v_2)$ *and* $\sigma_1(v_1) < \mu_2(u_2, v_2)$,

$\sigma_1(u_1) < \mu_2(u_2, v_2)$ *and* $\sigma_1(v_1) = \mu_2(u_2, v_2)$ and

$\sigma_1(u_1) = \mu_2(u_2, v_2)$ *and* $\sigma_1(v_1) = \mu_2(u_2, v_2)$.

Evidently , this case can be proved by the above three cases.

Hence, if all the arcs of G_1 and G_2 are strong then all the arcs of G_1 X G_2 are strong.

Example for case (i)

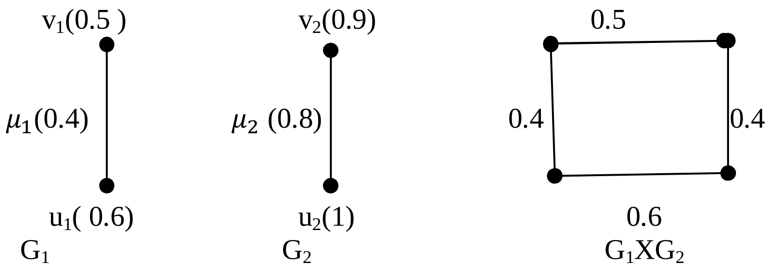

Figure 7:

Example for Case (ii)

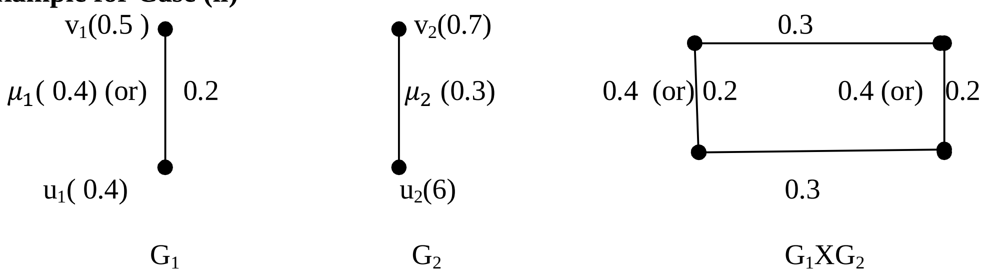

Figure 8:

Example for Case (iii)

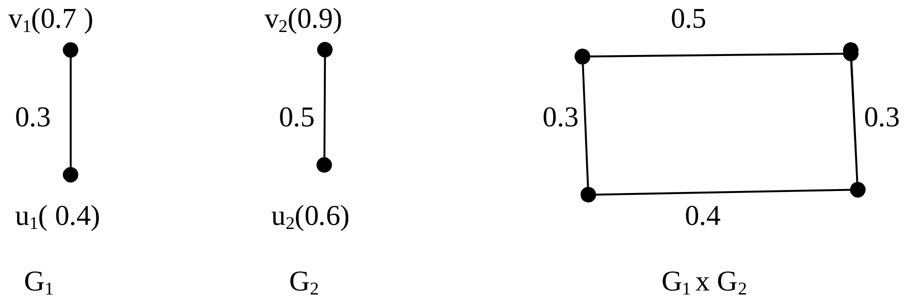

Figure 9:

Examples of the above three cases shows that if all the arcs of G_1 and G_2 are strong then all the arcs of G_1 X G_2 are strong.

Note 3.5. Converse need not be true.
If all the arcs in G_1 X G_2 are strong, it is not necessary that all arcs in G_1 or in G_2 must be strong. In other words, non strong arc can exist in G_1 or in G_2.

Note 3.6. If either G_1 or G_2 have an non strong arc then G_1 X G_2 need not have non strong arc.
But, if both have non strong arc then G_1 X G_2 must have non strong arc.
The following theorem proved that existence of non strong in G_1 X G_2.

Theorem 3.7. Let G_1 and G_2 be two fuzzy graph of which at least one has non strong arc, then the existing non strong arc in G_1 X G_2 depends on the fuzzy value of the vertices of G_1 or G_2 depends upon the non strong arc that appears in G_2 or in G_1 respectively.
That is, though G_1 or G2 have non strong arc, it is not necessary that G_1 X G_2 must have non strong arc.
Proof: To prove this hypothesis, we can take the minimum consideration of fuzzy graph as follows.
Let $G_1(\sigma_1, \mu_1)$ be fuzzy graph which has one non strong and let $G_2(\sigma_2, \mu_2)$ be another fuzzy graph which has no non strong respectively in fig (v) (a) and (b)

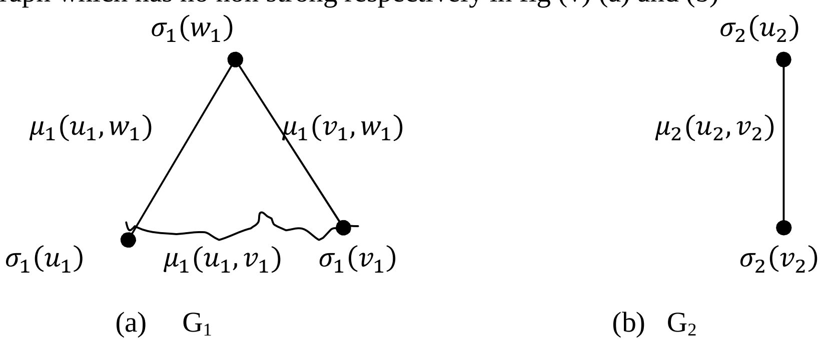

Figure 10:

Assume that $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$. Therefore , $\mu_1(u_1,v_1)$ is a non strong arc G_1(by theorem 3.1), there is no non strong in G_2
Claim: The existing non strong $\mu_1(u_1,v_1)$ arc in G_1 X G_2 depends on $\sigma_2(u_2)$ and $\sigma_2(v_2)$
The proof of this claim to be discussed in three cases.

Case 1: Let $\mu_1(u_1,v_1) < \sigma_2(u_2)$ and $\mu_1(u_1,v_1) < \sigma_2(v_2)$ (i)

Let G($\sigma_1\ X\ \sigma_2,\ \mu_1\ X\ \mu_2$) be the Cartesian product of fuzzy graph $G_1(\sigma_1,\mu_1)$ and $G_2(\sigma_2,\mu_2)$ are drawn in Fig 11.

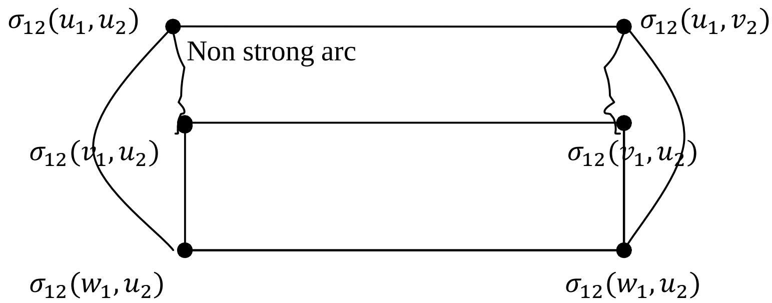

Figure 11:

Now, fuzzy value of the arc of G_1 X G_2 are found as follows.
μ_1X$\mu_2\{(u_1,u_2),(v_1,u_2)\}$ = min{ $\sigma_2(u_2),\mu_1(u_1,v_1)\}$ = $\mu_1(u_1,v_1)$ (by (i))
μ_1Xμ_2 $\{(v_1,u_2),(w_1,u_2)\}$ = min { $\sigma_2(u_2),\mu_1(v_1,w_1)\}$ = $\mu_1(v_1,w_1)$ (or) $\sigma_2(u_2)$
(Since, $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$ and by(i))
μ_1Xμ_2 $\{(u_1,u_2),(w_1,u_2)\}$ = min { $\mu_1(u_1,w_1),\ \sigma_2(u_2)\}$ = $\mu_1(u_1,w_1)$ (or) $\sigma_2(u_2)$.

These three fuzzy arc formed in one of the fuzzy cycle in G_1 X G_2 as in fig (v), in which $\mu_1(u_1,v_1)$ is the lowest value and this becomes non strong , since $\mu_1(u_1,v_1) < \sigma_2(u_2)$.

Hence , there is one non strong exist in G_1 X G_2 when non strong $\mu_1(u_1,v_1)$ arc in G_1 depends on $\sigma_2(u_2)$ such that $\mu_1(u_1,v_1) < \sigma_2(u_2)$.
Similarly, μ_1X$\mu_2\{(u_1,v_2),(v_1,v_2)\}$ = min{ $\sigma_2(v_2),\mu_1(u_1,v_1)\}$ = $\mu_1(u_1,v_1)$ (by (i))
μ_1Xμ_2 $\{(v_1,v_2),(w_1,v_2)\}$ = min { $\sigma_2(v_2),\mu_1(v_1,w_1)\}$ = $\mu_1(v_1,w_1)$ (or) $\sigma_2(v_2)$
(Since, $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$ and by(i))
μ_1Xμ_2 $\{(u_1,v_2),(w_1,v_2)\}$ = min { $\mu_1(u_1,w_1),\ \sigma_2(v_2)\}$ = $\mu_1(u_1,w_1)$ (or) $\sigma_2(v_2)$

These three fuzzy arc forms another fuzzy cycle in G_1 X G_2 as in fig (vi), in which $\mu_1(u_1,v_1)$ is the lowest value and becomes non strong arc, since $\mu_1(u_1,v_1) < \sigma_2(v_2)$.

From the above argument , it is concluded that the non strong arc existing in G_1 X G_2 depends on $\sigma_2(u_2)$ and $\sigma_2(v_2)$ of G_2 where non strong $\mu_1(u_1,v_1)$ arc is in G_1 and therefore there are two non strong arcs in G_1 X G_2 ,since non strong arc $\mu_1(u_1,v_1) < \sigma_2(u_2)$ and $\mu_1(u_1,v_1) < \sigma_2(v_2)$.

Case 2: Let $\mu_1(u_1,v_1) > \sigma_2(u_2)$ and $\mu_1(u_1,v_1) > \sigma_2(v_2)$ (ii)

Now, fuzzy value of the arc of G_1 X G_2 are found as follws.
μ_1X$\mu_2\{(u_1,u_2),(v_1,u_2)\}$ = min{ $\sigma_2(u_2),\mu_1(u_1,v_1)\}$ = $\sigma_2(u_2)$ (by (ii))

μ_1Xμ_2 $\{(v_1,u_2),(w_1,u_2)\}$ = min { $\sigma_2(u_2),\mu_1(v_1,w_1)\}$ = $\sigma_2(u_2)$
(Since, $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$ and by(i))
μ_1Xμ_2 $\{(u_1,u_2),(w_1,u_2)\}$ = min { $\mu_1(u_1,w_1),\ \sigma_2(u_2)\} = \mu_1(u_1,w_1)$.

These three fuzzy arc forms fuzzy cycle in G_1 X G_2 as in fig (vi), in which all the arc value becomes strong since $\mu_1(u_1,v_1) > \sigma_2(u_2)$ and $\sigma_2(u_2)$ is the lowest value that occurs more than once in G_1 X G_2. Similarly , the same result provided by non strong arc $\mu_1(u_1,v_1)$ in $G_1$by the Cartesian product with $\sigma_2(v_2)$.

Therefore, in this case no non strong arc exist in G_1 X G_2 , since $\mu_1(u_1,v_1) > \sigma_2(u_2)$ and $\mu_1(u_1,v_1) > \sigma_2(v_2)$

Case 3: Let $\mu_1(u_1,v_1) = \sigma_2(u_2)$ and $\mu_1(u_1,v_1) = \sigma_2(v_2)$

Now, fuzzy value of the arc of G_1 X G_2 are found as follows.
μ_1X$\mu_2$$\{(u_1,u_2),(v_1,u_2)\}$ = min{ $\sigma_2(u_2),\mu_1(u_1,v_1)\} = \sigma_2(u_2) = \mu_1(u_1,v_1)$ (by (ii))
μ_1Xμ_2 $\{(v_1,u_2),(w_1,u_2)\}$ = min { $\sigma_2(u_2),\mu_1(v_1,w_1)\}$ = $\sigma_2(u_2) = \mu_1(u_1,v_1)$
(Since , $\mu_1(u_1,v_1) < \mu_1(v_1,w_1) \leq \mu_1(u_1,w_1)$ and by(ii))
μ_1Xμ_2 $\{(u_1,u_2),(w_1,u_2)\}$ = min { $\mu_1(u_1,w_1),\ \sigma_2(u_2)\} = \sigma_2(u_2) = \mu_1(u_1,v_1)$.

These three arc with same fuzzy value in fuzzy cycle in G_1 X G_2 are strong (by theorem 3.2), similarly the same result exist for other three arcs.

Therefore, in this case also no non strong arc exist in G_1 X G_2 ,

Case 4: Let $\mu_1(u_1,v_1) < \sigma_2(u_2)$ and $\mu_1(u_1,v_1) \geq \sigma_2(v_2)$

From the above cases, in case (1),there is one non strong arc existing in G_1 X G_2 when $\mu_1(u_1,v_1) < \sigma_2(u_2)$ and in case (2), case (3) and case (4) , all the arcs of G_1 X G_2 are strong, when $\mu_1(u_1,v_1) \geq \sigma_2(v_2)$.

From the above four cases ,it is concluded that non strong arc exist in G_1 X G_2 only when fuzzy value of non strong arc of G_1 (or G_2) must be greater than the degree value of the vertex of G_2 (or G_1) respectively. Hence ,the existing non strong arc in G_1 X G_2 depends on the fuzzy value of the vertices of G_1 or G_2 depending upon the non strong arc that appears in G_2 or in G_1 respectively.

Corollary 3.8. The number of non strong arc of G_1 X G_2 = m_1 P_2 + m_2P_1
where,

- **m_1 -** number of vertex of G_1 ,whose fuzzy value must greater than fuzzy value of non strong arc in other fuzzy graph G_2
- **m_2 -** number of vertex of G_2 , whose fuzzy value must greater than fuzzy value of non strong arc in other fuzzy graph G_1
- **P_1 -** number of non strong arc in G_1
- **P_2 -** number of non strong arc in G_2

4. Conclusion

In this paper, we discussed the condition for non strong arc to appear in Cartesian product of fuzzy graph, fuzzy path and fuzzy cycle etc. and deduced the formula for finding non strong arc in Cartesian product of fuzzy graph. Next our aim is to find the domination number of Cartesian product fuzzy graph and various type fuzzy graph by using strong arc.

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