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The Diophantine Equation $x^4 + y^2 = z^4$ is Insolvable in Positive Integers when at Least One of x, y is Prime

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Abstract. In this paper, we consider solutions in positive integers x, y, z of the title equation. It is established that the equation has no solutions in the following two cases: (i) When $x \geq 2$ and $y \geq 2$ are two distinct primes. (ii) When either one of x, y is prime, and the other one is composite. We presume that when x and y are composites, the equation has no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 3, 4]. The title equation stems from the equation $p^x + q^y = z^2$.

Whereas in most articles, the values x, y are investigated for the solutions of the equation, in this paper these values are fixed positive integers. In the equation

$$x^4 + y^2 = z^4 \tag{1}$$

we consider the case when $x \geq 2$ and $y \geq 2$ are two distinct primes, and also the case when one of x, y is prime, and the other is composite. This is respectively done in the following Sections 2 and 3.

2. The equation $x^4 + y^2 = z^4$ when x and y are primes

In this section, we investigate the solutions of $x^4 + y^2 = z^4$ when x, y are two distinct primes. The result is contained in the following Theorem 2.1.

Theorem 2.1. Suppose that $x \geq 2$, $y \geq 2$ are two distinct primes. Then the equation $x^4 + y^2 = z^4$ has no solutions.

Proof: All the possibilities of the equation are as follows:

(a) $x = 2$ and y is an odd prime.
(b) $y = 2$ and x is an odd prime.
(c) x, y are odd primes.

(a) Suppose that $x = 2$ and y is an odd prime.
Equation (1) yields

$$16 + y^2 = z^4, \qquad z \text{ is odd.} \tag{2}$$

From (2) we have

$$y^2 = z^4 - 16 = (z^2 - 4)(z^2 + 4) = (z - 2)(z + 2)(z^2 + 4). \tag{3}$$

Since y is prime, the three divisors of y^2 namely $1, y, y^2$ can never satisfy the three factors in (3). Hence, **(a)** does not exist, and $16 + y^2 = z^4$ has no solutions.

(b) Suppose that $y = 2$ and x is an odd prime.
Equation (1) implies

$$x^4 + 4 = z^4, \qquad z \text{ is odd.}$$

Hence

$$4 = z^4 - x^4 = (z - x)(z^3 + z^2x + zx^2 + x^3). \tag{4}$$

For all odd primes x, and since z is odd, both factors on the right of equation (4) are even. Equation (4) is then clearly impossible, and $x^4 + 4 = z^4$ has no solutions.

(c) Suppose that x, y are odd primes.
Equation (1) yields

$$x^4 = z^4 - y^2 = (z^2 - y)(z^2 + y), \qquad z \text{ is even.} \tag{5}$$

Equation (5) has only two possibilities (three other possibilities are a priori eliminated), namely:
(i) $z^2 - y = 1, \quad z^2 + y = x^4.$
(ii) $z^2 - y = x, \quad z^2 + y = x^3.$

If (i) exists, then $z^2 = y + 1$ and $2y + 1 = x^4$. Each prime x is either of the form $4N + 1$ or of the form $4N + 3$. In any case, x^4 is of the form $4M + 1$, implying that the equality $2y + 1 = 4M + 1$ or $y = 2M$ is impossible since y is an odd prime. Thus, (i) does not exist.

If (ii) exists, then $z^2 = y + x$ implies that

$$z^2 + y = (y + x) + y = 2y + x = x^3 \quad \text{or} \quad 2y = x(x^2 - 1)$$

which is impossible since x, y are odd primes and $x \nmid y$. Hence, (ii) does not exist, and case **(c)** is complete.

The equation $x^4 + y^2 = z^4$ has no solutions.

This concludes the proof of **Theorem 2.1**. □

3. The equation $x^4 + y^2 = z^4$ when either x or y is prime

In this section, we consider all the possibilities of $x^4 + y^2 = z^4$ when x is prime and y is composite or vice versa. This is done in the following Theorem 3.1.

Theorem 3.1. Suppose that x, y are positive integers. If one of x, y is prime and the other is composite, then $x^4 + y^2 = z^4$ has no solutions.

Proof: All the possibilities of the equation are as follows:

Case 1. x is an odd prime, y is composite.
Case 2. $x = 2$, y is composite.
Case 3. x is composite, $y = 2$.
Case 4. x is composite, y is an odd prime.

Each of the four cases is self-contained, and considered separately.

Case 1. Suppose x is an odd prime, and y is composite.

From (1) we have

$$x^4 = z^4 - y^2 = (z^2 - y)(z^2 + y). \tag{6}$$

Since x is prime, equation (6) has only two possibilities (three other possibilities are a priori eliminated), namely:

(i) $z^2 - y = 1$, $z^2 + y = x^4$.
(ii) $z^2 - y = x$, $z^2 + y = x^3$.

Suppose (i), i.e., $z^2 - y = 1$ and $z^2 + y = x^4$.
We shall assume that there exist values x, y, z satisfying both equations simultaneously, and reach a contradiction.

Our assumption implies that $y = z^2 - 1$ and $y = x^4 - z^2$. Hence, $z^2 - 1 = x^4 - z^2$ or

$$2z^2 - x^4 = 1. \tag{7}$$

For the smallest odd prime $x = 3$, min $z = 7$ and $2z^2 - x^4 = 2\cdot 7^2 - 3^4 = 17$. It is easily verified for each and every prime $x > 3$ that $\min(2z^2 - x^4) > 17$. Thus, equation (7) never exists, and the contradiction derived implies that our assumption is false. Case (i) is complete.

Suppose (ii), i.e., $z^2 - y = x$ and $z^2 + y = x^3$.
Adding both equations results in

$$2z^2 = x + x^3 = x(x^2 + 1). \tag{8}$$

Since x is an odd prime, it follows from (8) that $x \mid z$. Denote $z = Ax$ where A is a positive integer, and $2z^2 = 2A^2x^2$. But, this implies that (8) is impossible since the left side of (8) is then a multiple of x^2, whereas the right side of (8) is a multiple of x only.
Case (ii) does not exist, and **Case 1** is complete.

The equation $x^4 + y^2 = z^4$ has no solutions.

Case 2. Suppose that $x = 2$, and y is composite.

From (1) we have

$$2^4 = z^4 - y^2 = (z^2 - y)(z^2 + y). \tag{9}$$

Equation (9) has only two possibilities (three other possibilities are a priori eliminated). These are:

(i) $z^2 - y = 1$, $z^2 + y = 16$.
(ii) $z^2 - y = 2$, $z^2 + y = 8$.

In (i) $z^2 = y + 1$, but $z^2 + y = 2y + 1 \neq 16$. Thus, (i) is impossible. For (ii), we have $z^2 = y + 2$ or $z^2 + y = 2y + 2 = 8$ and $y = 3$. But, $y = 3$ is not composite ($y = 3$ implies that z is not an integer). Hence, (ii) is impossible. This concludes **Case 2**.

The equation $2^4 + y^2 = z^4$ has no solutions.

Case 3. Suppose x is composite, and $y = 2$.

We have from (1)

$$2^2 = z^4 - x^4 = (z - x)(z^3 + z^2x + zx^2 + x^3). \tag{10}$$

For all odd values x, and for all even values x, both factors on the right side of equation (10) are even. Equation (10) is then impossible. Thus **Case 3** does not exist.

The equation $x^4 + 2^2 = z^4$ has no solutions.

Case 4. Suppose x is composite, and y is an odd prime.

From (1) we have

$$y^2 = z^4 - x^4 = (z^2 - x^2)(z^2 + x^2) = (z - x)(z + x)(z^2 + x^2). \tag{11}$$

Since y is prime, the three divisors of y^2 being 1, y, y^2 never satisfy the three factors on the right side of (11). Thus (11) is impossible, and **Case 4** does not exist.

The equation $x^4 + y^2 = z^4$ has no solutions.

The proof of **Theorem 3.1** is complete. □

Final remark. We presume that when x and y are composites, then the equation $x^4 + y^2 = z^4$ has no solutions.

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