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On Solutions of the Diophantine Equations $p^4 + q^4 = z^2$ and $p^4 - q^4 = z^2$ when p and q are Primes

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Abstract. In this article, we consider the equations $p^4 + q^4 = z^2$ and $p^4 - q^4 = z^2$ when p, q are primes and z is a positive integer. We establish that $p^4 + q^4 = z^2$ has no solutions for all primes $2 \le p < q$. For $p^4 - q^4 = z^2$ when $2 \le q < p$, it is shown: (i) For $q = 2$ the equation has no solutions. (ii) The equation $z^2 = p^4 – q^4 = (p^2 – q^2)(p^2 + q^2)$ is impossible when each factor is equal to a square. (iii) When each factor is not a square, conditions which must be satisfied simultaneously are determined for p^2 and q^2. For all primes $p < 2100$, these conditions are not fulfilled simultaneously. It is conjectured for all primes $p > 2100$ and $q > 2$ that the equation has no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are [2, 3, 4] and others.

In this paper, we consider the two equations

$$p^4 + q^4 = z^2,$$
$$p^4 - q^4 = z^2$$

where p, q are primes, and z is a positive integer. All other values denote positive integers too.

In the two self-contained Sections 2 and 3, the two equations are respectively investigated for solutions. In Section 2, we show for all primes $2 \le p < q$ that $p^4 + q^4 = z^2$ has no solutions. In Section 3, we consider $p^4 - q^4 = z^2$, and prove that when $q = 2$ and when $2 < q < p < 2100$ the equation has no solutions. In our proof we use "Pythagorean triples".

2. Solutions of the equation $p^4 + q^4 = z^2$

For all primes p, q, it is shown in the following theorem that $p^4 + q^4 = z^2$ has no solutions. Without loss of generality, we shall assume that $p < q$.

Theorem 2.1. Suppose that $2 \le p < q$ are primes. Then the equation $p^4 + q^4 = z^2$ has no solutions.

Proof: First we consider the case $p = 2$, and then the case $p > 2$.

Suppose $p = 2$. We have

$$16 + q^4 = z^2, \qquad z \text{ is odd.} \tag{1}$$

From (1) it follows that $z^2 - q^4 = 16$ or

$$(z - q^2)(z + q^2) = 2^4. \tag{2}$$

The five divisors of 2^4 are 1, 2^1, 2^2, 2^3, 2^4. Thus from (2) we obtain

$$z - q^2 = 1 \quad \text{and} \quad z + q^2 = 2^4, \qquad z - q^2 = 2^1 \quad \text{and} \quad z + q^2 = 2^3,$$

where the last three possibilities are a priori eliminated.

The case $z - q^2 = 1$ and $z + q^2 = 16$ is a priori impossible, since the difference of two odd integers z and q^2 is never equal to 1. Hence, this case does not exist.

If $z - q^2 = 2$ and $z + q^2 = 8$, then $z = q^2 + 2$ or $2q^2 + 2 = 8$ and $q^2 = 3$ is impossible.

Thus, $p \ne 2$ and $16 + q^4 = z^2$ has no solutions.

Suppose $p > 2$. The equation $p^4 + q^4 = z^2$ yields

$$p^4 = z^2 - q^4 = (z - q^2)(z + q^2). \tag{3}$$

The five divisors of p^4 are 1, p^1, p^2, p^3, p^4. Hence, from (3) we have

$$z - q^2 = 1 \quad \text{and} \quad z + q^2 = p^4, \qquad z - q^2 = p^1 \quad \text{and} \quad z + q^2 = p^3,$$

where the last three possibilities are a priori eliminated.

If $z - q^2 = 1$ and $z + q^2 = p^4$, then $z = q^2 + 1$ and $2q^2 + 1 = p^4$. Hence,

$$2q^2 = p^4 - 1 = p^4 - 1^4 = (p - 1)(p^3 + p^2 + p^1 + 1). \tag{4}$$

The right side of (4) is clearly a multiple of at least 4, whereas $2q^2$ is a multiple of exactly 2. This contradiction implies that the first possibility is impossible.

If $z - q^2 = p$ and $z + q^2 = p^3$, then $z = q^2 + p$ and $2q^2 + p = p^3$. Since gcd $(p, q) = 1$, it follows that $2q^2 + p = p^3$ does not exist, and this possibility is also impossible. Hence, when $p > 2$, $p^4 + q^4 = z^2$ has no solutions as asserted.

This concludes the proof of **Theorem 2.1**. □

3. Solutions of the equation $p^4 - q^4 = z^2$

In this section we establish: First, for $q = 2$ the equation has no solutions. Secondly, for all primes $q > 2$ and $p < 2100$, the equation also has no solutions.

Theorem 3.1. Suppose that $2 \leq q < p$ are primes. Then the equation $p^4 - q^4 = z^2$ has no solutions

(i) when $2 = q$,

(ii) when $2 < q < p < 2100$.

Proof: (i) When $q = 2$, we have

$$p^4 - 16 = z^2, \qquad z \text{ is odd.} \tag{5}$$

From (5) it follows that $p^4 - z^2 = 16$ or

$$(p^2 - z)(p^2 + z) = 2^4. \tag{6}$$

The five divisors of 2^4 are $1, 2^1, 2^2, 2^3, 2^4$. Thus from (6) we have

$$p^2 - z = 1 \quad \text{and} \quad p^2 + z = 2^4, \qquad p^2 - z = 2^1 \quad \text{and} \quad p^2 + z = 2^3,$$

where the last three possibilities are a priori eliminated.

The case $p^2 - z = 1$ and $p^2 + z = 16$ is a priori impossible, since the difference of two odd integers p^2 and z is never equal to 1. The first possibility does not exist.

If $p^2 - z = 2$ and $p^2 + z = 8$, then $p^2 = z + 2$ or $2z + 2 = 8$ and $z = 3$. But, when $z = 3$ then $p^2 = 5$ which is impossible. Hence, the second possibility does not exist.

Thus $q \neq 2$, and $p^4 - 16 = z^2$ has no solutions. Case (i) is complete.

When $q < p$, then $p^4 - q^4 = z^2$ yields

$$p^4 - q^4 = (p^2 - q^2)(p^2 + q^2) = z^2. \tag{7}$$

Let $R > 1$ be an integer which is not a square. Equation (7) has two possibilities, namely

(a) $p^2 - q^2 = A^2$ and $p^2 + q^2 = B^2$,

(b) $p^2 - q^2 = C^2 \cdot R$ and $p^2 + q^2 = D^2 \cdot R$.

(a) Suppose that $p^2 - q^2 = A^2$ and $p^2 + q^2 = B^2$.

The values A^2, B^2 are even, hence A, B are even. Denote $A^2 = 4N$ and $B^2 = 4M$. Thus, the sum of the two equalities in our supposition yields

$$2p^2 = A^2 + B^2 = 4N + 4M = 4(N + M)$$

which is impossible since p is an odd prime.

Case **(a)** does not exist.

(b) Suppose that $p^2 - q^2 = C^2R$ and $p^2 + q^2 = D^2R$.

The sum of the equalities in our supposition yields

$$2p^2 = R(C^2 + D^2). \tag{8}$$

From (8) it follows that $R \mid 2p^2$, and since $\gcd(R, p) = 1$, therefore $R = 2$. Hence

$$p^2 - q^2 = 2C^2 \quad \text{and} \quad p^2 + q^2 = 2D^2,$$

from which

$$p^2 = C^2 + D^2 \quad \text{and} \quad q^2 = D^2 - C^2.$$

Since $q^2 = D^2 - C^2 = (D - C)(D + C)$ and q is prime, it follows that $D - C = 1$ and $D + C = q^2$ implying that $q^2 = 2C + 1$. The value $q^2 - 1 = (q - 1)(q + 1) = 2C$ yields for all primes q that $C = 4T$. We have $p^2 - q^2 = 2C^2$ or $p^2 = 2C^2 + q^2 = 2C^2 + (2C + 1) = C^2 + (C + 1)^2$. Thus,

$$q^2 = C + (C + 1), \qquad p^2 = C^2 + (C + 1)^2, \qquad C = 4T. \tag{9}$$

The equalities in (9) must be satisfied simultaneously with primes p, q.

Suppose (ii) $2 < q < p < 2100$.

In (9), the equality $C^2 + (C + 1)^2 = p^2$ suggests the use of "Pythagorean triples" (abbreviated triples) denoted (a, b, c) if $a^2 + b^2 = c^2$. All the triples of the form $C^2 + (C + 1)^2 = p^2$ where $p < 2100$ have been examined in [5]. There is exactly one triple of this form which satisfies only two of the three conditions in (9). The triple is demonstrated in Table 1. Hence, for all primes $2 < q < p < 2100$ the three conditions in (9) are not met simultaneously, and $p^4 - q^4 = z^2$ has no solutions.

This completes part (ii), and **Theorem 3.1**. □

For the convenience of the reader, values C, q^2, p^2 are presented in the following Table 1.

Table 1. Values of C, q^2 and p^2

T	$C = 4T$	$C + 1 = 4T + 1$	$C + (C + 1) = q^2$	$C^2 + (C + 1)^2 = p^2$
1	$C = 4$	$C + 1 = 5$	$4 + 5 = 3^2 = q^2$	$4^2 + 5^2 = 41 \neq p^2$
3	$C = 12$	$C + 1 = 13$	$12 + 13 = 5^2 = q^2$	$12^2 + 13^2 = 313 \neq p^2$
6	$C = 24$	$C + 1 = 25$	$24 + 25 = 7^2 = q^2$	$24^2 + 25^2 = 1201 \neq p^2$
15	$C = 60$	$C + 1 = 61$	$60 + 61 = 11^2 = q^2$	$60^2 + 61^2 = 7321 \neq p^2$
21	$C = 84$	$C + 1 = 85$	$84 + 85 = 13^2 = q^2$	$84^2 + 85^2 = 14281 \neq p^2$
5	$C = 20$	$C + 1 = 21$	$20 + 21 = 41 \neq q^2$	$20^2 + 21^2 = 29^2 = p^2$

Remark 3.1. It is needless to say, that a priori it seems impossible that the three conditions in (9) may be satisfied simultaneously.
The validity of this statement was verified for all primes $p < 2100$.

We can now sum up the results for both equations as follows:

Conjecture 1. For all primes $p > 2100$ and $q > 2$, the equation $p^4 - q^4 = z^2$ has no solutions.

If **Conjecture 1** is indeed true, then both equations $p^4 + q^4 = z^2$ and $p^4 - q^4 = z^2$ have no solutions.

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