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A Type of the Cauchy-Euler Equations: Distinct Complex Roots

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Abstract. In this research, the authors give the family of all homogeneous Cauchy-Euler equations such that each equation has general solution depending on distinct complex numbers and their conjugates.

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1. Introduction

Consider a homogeneous Cauchy-Euler equation of order n of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dy^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = 0. \tag{1.1}$$

where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. Details for methods to find solutions of the equation (1.1) was explained in [2, 3, 4, 9]. Moreover, Sabuwala and Leon [7] studied the particular solution for the most general *n*-th order Euler differential equation when the non-homogeneity is a polynomial. They found a formula which can be used to compute the unknown coefficients in the form of the particular solution. It is well known that the general solution of (1.1) can be found from the characteristic equation

$$\sum_{j=1}^{n} a_j \prod_{i=1}^{j} (m - i + 1) + a_0 = 0 \tag{1.2}$$

of the linear ordinary differential equation with constant coefficients

$$\left(\sum_{j=1}^{n} a_j \prod_{i=1}^{j} \left(\frac{d}{dt} - i + 1 \right) + a_0 \right) y = 0,$$

where $t = \ln x$. In general, the general solution of any homogeneous Cauchy-Euler equations depends on zeros of the polynomial $\sum_{j=1}^{n} a_j \prod_{i=1}^{j} (m - i + 1) + a_0.$

The aim of this paper is to give the family of all Cauchy-Euler equations (1.1) such that

$$y=\sum_{j=1}^{k}x^{\alpha_j}\left(c_{2j-1}\sin(\beta_j\ln x)+c_{2j}\cos(\beta_j\ln x)\right)$$

is the general solution of (1.1) on $(0,\infty)$ where α_j and β_j are real and imaginary parts of distinct z_j with $\beta_j\neq 0$ for all $j=1,...,k$.

2. Preliminary

In this section, we shall give the related basic notions that can be found in [1, 5, 8].

Let $n\in\mathbb{N}$, $a_0,a_1,...,a_n\in\mathbb{R}$ with $a_n\neq 0.$ An ordinary differential equation of the form

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dy^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=0 \tag{2.1}$$

is said to be a *homogeneous linear ordinary differential equation* with constant coefficients. By a transformation $y=e^{mx}$, where m is a suitable number, the equation (2.1) is transformed into the polynomial equation

$$a_nm^n+a_{n-1}m^{n-1}+...+a_1m+a_0=0,$$

which is said to be the *characteristic equation* of (2.1).

A linear ordinary differential equation form

$$a_nx^n\frac{d^ny}{dx^n}+a_{n-1}x^{n-1}\frac{d^{n-1}y}{dy^{n-1}}+\cdots+a_1x\frac{dy}{dx}+a_0y=0 \tag{2.2}$$

is called a *homogeneous Cauchy-Euler equation*.

Theorem 2.1. [5] Let k be a positive integer, $n=2k$ and $\alpha_1,\alpha_2,...,\alpha_k,\beta_1,\beta_2,...,\beta_k,a_0,a_1,...,a_{2k}$ be real numbers with $a_n\neq 0.$ Then the distinct complex numbers $z_1=\alpha_1+\beta_1 i,\ \ z_2=\alpha_2+\beta_2 i,...,z_k=\alpha_k+\beta_k i,\ z_{k+1}=\overline{z_1},\ \ z_{k+2}=\overline{z_2},...,\ z_{2k}=\overline{z_k}$ are distinct $2k$ zeroes of the polynomial $\left(\sum_{j=1}^{n}a_j\prod_{i=1}^{j}\left(\frac{d}{dt}-i+1\right)+a_0\right)y=0$ if and only if

$$y=\sum_{j=1}^{k}x^{\alpha_j}\left(c_{2j-1}\sin(\beta_j\ln x)+c_{2j}\cos(\beta_j\ln x)\right) \tag{2.3}$$

is the general solution of homogeneous Cauchy-Euler equation (2.2) on an open interval $(0,\infty)$, where $c_1,c_2,...,c_{2k}$ are arbitrary constants.

Definition 2.1. [6] For each $j,k\in\mathbb{N}$ with $j\leq k$ we define

$$N_k:=\{1,2,...,k\},$$

$$P_{j,k}:=\{a_1a_2\cdots a_j : a_1,a_2,...,a_j\in N_k \text{ and } a_1<a_2<\cdots<a_j\},$$

$$N_{j,k}:=\sum_{p\in P_{j,k}}p,\quad N_{0,0}:=1\quad\text{and}\quad N_{0,k}:=1,$$

and for every integers j,k with $k > j$ we define $N_{k,j} := 0.$

Lemma 2.1. [6] Let $n \in \mathbb{N}$ with $n \geq 2,$ $m \in \mathbb{C}$ and $a_1,...,a_n \in \mathbb{R}$ with $a_n \neq 0.$ Then

$$\sum_{j=1}^{n} a_j \prod_{i=1}^{j} (m-i+1) + a_0 = a_n m^n + \sum_{j=1}^{n-1} m^{n-j} \sum_{i=0}^{j} (-1)^i N_{i,n+i-j-1} a_{n+i-j} + a_0. \qquad (2.4)$$

The following theorem is a direct consequence of Theorem 2.1 and Lemma 2.1.

Theorem 2.2. Let k be a positive integer, $n = 2k$ and $\alpha_1, \alpha_2, ..., \alpha_k, \beta_1, \beta_2, ..., \beta_k,$ $a_0, a_1, ..., a_{2k}$ be real numbers with $a_n \neq 0.$ Then the distinct complex numbers $z_1 = \alpha_1 + \beta_1 i,$ $z_2 = \alpha_2 + \beta_2 i, ..., z_k = \alpha_k + \beta_k i,$ $z_{k+1} = \overline{z_1},$ $z_{k+2} = \overline{z_2}, ...,$ $z_{2k} = \overline{z_k}$ are distinct $2k$ zeroes of the polynomial $a_n m^n + \sum_{j=1}^{n-1} m^{n-j} \sum_{i=0}^{j} (-1)^i N_{i,n+i-j-1} a_{n+i-j} + a_0$ if and only if (2.3) is the general solution of homogeneous Cauchy-Euler equation (2.2) on an open interval $(0,\infty),$ where $c_1, c_2, ..., c_{2k}$ are arbitrary constants.

3. Main theorems

In this section, before proceeding to our main results, the following terminologies and concepts are required.

Definition 3.1. Let n be a positive integer and $z_1, z_2, ..., z_n$ be distinct complex numbers. Then for every integer j,k with $1 \leq j \leq k \leq n$ we define

$$C_k = \{z_1, z_2, ..., z_k\},$$

$$S_{j,k} = \{\{a_1, a_2, ..., a_j\} : a_1, a_2, ..., a_j \in C_k \text{ and } a_1, a_2, ..., a_j \text{ are distinct}\},$$

$$C_{j,k} = \sum_{\{a_1,...,a_j\} \in S_{j,k}} a_1 a_2 \cdots a_j, \quad C_{0,0} = C_{0,j} = 1,$$

and $C_{k,j} = 0$ for all $j < k \leq n.$

Form above definition, it is important to note that

$$C_{k,k} = z_1 z_2 \cdots z_k \text{ and } C_{k,k} = C_{k+1,k+1}$$

for every positive integer $k.$ Furthermore, we have the following applicable lemma.

Lemma 3.1. For every positive n. If $z_1, z_2, ..., z_n$ are distinct complex numbers and $C_s = \{z_1, z_2, ..., z_s\}$ for all positive integer s with $s \leq n$, then for every positive integer i, j with $i \leq j \leq n,$

$$C_{i,j-1} + z_j C_{i-1,j-1} = C_{i,j}. \qquad (3.1)$$

Proof: We shall proof by mathematical induction on n. For $n=2$, let i, j be positive integers such that i, j with $i \le j \le 2$ and z_1, z_2 are complex numbers and $C_2 = \{z_1, z_2\}$. Then we have 3 cases: $i = j = 1$, $i = 1$ and $j = 2$, and $i = j = 2$. It follow that $C_{1,0} + z_1 C_{1-1,1-1} = z_1 = C_{1,1}$, $C_{1,1} + z_2 C_{1-1,2-1} = z_1 + z_2 = C_{1,2}$ and $C_{2,1} + z_2 C_{2-1,2-1} = z_1 \cdot z_2 = C_{2,2}$, this implies the lemma is true for $n=2$.

Next, we let k be arbitrary positive integer with $k \ge 2$. Suppose that this lemma is true for $n=k$, that is for every distinct complex numbers $z_1, z_2, ..., z_k$ and $C_s = \{z_1, z_2, ..., z_s\}$ for all positive integer s with $s \le n$, for every positive integers i, j with $i \le j \le k$, $C_{i,j-1} + z_j C_{i-1,j-1} = C_{i,j}$. Let $z_1, z_2, ..., z_{k+1}$ be distinct complex numbers and $C_s = \{z_1, z_2, ..., z_s\}$ for all positive integer s with $s \le k+1$. Let i, j be positive integers with $i \le j \le k+1$. We shall proof that

$$C_{i,j-1} + z_j C_{i-1,j-1} = C_{i,j}.$$

Since $z_1, z_2, ..., z_k$ are complex numbers, from the inductive hypothesis, we obtain

$$C_{i,j-1} + z_j C_{i-1,j-1} = C_{i,j}.$$

for every positive integers i, j with $i \le j \le k$, and so we only prove that

$$C_{i,j-1} + z_j C_{i-1,j-1} = C_{i,j}.$$

for all $i \le k+1$. Let i be a positive integer with $i \le k+1$ and let

$$R_{i,k+1} = \{\{a_1, a_2, ..., a_{i-1}, z_{k+1}\} : a_1, a_2, ..., a_{i-1} \in C_k \text{ and } a_1, a_2, ..., a_{i-1}, z_{k+1} \text{ are distinct}\}.$$

We claim that $S_{i,k} \cap R_{i,k+1} = \emptyset$ and $S_{i,k} \cup R_{i,k+1} = S_{i,k+1}$. For $S_{i,k} \cap R_{i,k+1} = \emptyset$, suppose that $S_{i,k} \cap R_{i,k+1} \ne \emptyset$. Let $\{a_1, a_2, ..., a_i\} \in S_{i,k} \cap R_{i,k+1}$. Then $\{a_1, a_2, ..., a_i\} \in S_{i,k}$ and $\{a_1, a_2, ..., a_i\} \in R_{i,k+1}$, and thus $a_1, a_2, ..., a_i \in C_k$ and $a_1, a_2, ..., a_i$ are distinct and there exists $j \in N_i$ such that $a_j = z_{k+1}$. Since $a_s \in C_k$ for every $s \in N_i$, we obtain $a_s \ne z_{k+1}$ which is impossible. Hence $S_{i,k} \cap R_{i,k+1} = \emptyset$.

Next, we shall prove that $S_{i,k} \cup R_{i,k+1} = S_{i,k+1}$. Let $\{a_1, a_2, ..., a_i\} \in S_{i,k} \cup R_{i,k+1}$. Then $\{a_1, a_2, ..., a_i\} \in S_{i,k}$ or $\{a_1, a_2, ..., a_i\} \in R_{i,k+1}$.

Case 1. Suppose that $\{a_1, a_2, ..., a_i\} \in S_{i,k}$. Then $a_1, a_2, ..., a_i \in C_k$ and $a_1, a_2, ..., a_i$ are distinct. Since $C_k \subseteq C_{k+1}$, $a_1, a_2, ..., a_i \in C_{k+1}$. It follows that $\{a_1, a_2, ..., a_i\} \in S_{i,k+1}$.

Case 2. Suppose that $\{a_1, a_2, ..., a_i\} \in R_{i,k+1}$. Then $a_1, a_2, ..., a_{j-1}, a_{j+1}, ..., a_i \in C_k$ are distinct element in C_k and there exists $j \in N_i$ such that $a_j = z_{k+1}$ and thus $a_1, ..., a_i \in C_{k+1}$. Hence $\{a_1, a_2, ..., a_i\} \in S_{i,k+1}$.

From both cases, we obtain $S_{i,k} \cup R_{i,k+1} \subseteq S_{i,k+1}$.

Now we let $\{a_1, a_2, ..., a_i\} \in S_{i,k+1}$. Then $a_1, a_2, ..., a_i \in C_{k+1}$ and $a_1, a_2, ..., a_i$ are distinct.

Case 1. Suppose that there exists $j \in N_i$ such that $a_j = z_{k+1}$. Since $a_1, a_2, ..., a_i$ are distinct complex numbers. Then $\{a_1, a_2, ..., a_i\} \in R_{i,k+1}$ and so $\{a_1, a_2, ..., a_i\} \in S_{i,k} \cup R_{i,k+1}$.

Case 2. Suppose that $a_j \neq z_{k+1}$ for all $j \in N_i$. Since for every $j \in N_i$, $a_i \in C_{k+1}$, we obtain $a_i \in C_{k+1}$. Hence $\{a_1, a_2, ..., a_i\} \in S_{i,k}$ and thus $\{a_1, a_2, ..., a_i\} \in S_{i,k} \cup R_{i,k+1}$.

From both cases, we obtain $S_{i,k+1} \subseteq S_{i,k} \cup R_{i,k+1}$. It follows that $S_{i,k} \cup R_{i,k+1} = S_{i,k+1}$. Hence

$$C_{i,k} + z_{k+1}C_{i-1,k} = C_{i,k} + z_{k+1} \sum_{\{a_1,a_2,...,a_{i-1}\} \in S_{i-1,k}} a_1 a_2 \cdots a_{i-1}$$

$$= \sum_{\{a_1,a_2,...,a_i\} \in S_{i,k}} a_1 a_2 \cdots a_i + \sum_{\{a_1,a_2,...,a_{i-1}\} \in S_{i-1,k}} a_1 a_2 \cdots a_{i-1} z_{k+1}.$$

Let $z_{k+1}S_{i-1,k} = \{\{a_1, a_2, ..., a_{i-1}, z_{k+1}\} : \{a_1, a_2, ..., a_{i-1}\} \in S_{i-1,k}\}$. Then

$$C_{i,k} + z_{k+1}C_{i-1,k} = \sum_{\{a_1,a_2,...,a_i\} \in S_{i,k}} a_1 a_2 \cdots a_i + \sum_{\{a_1,a_2,...,a_{i-1},z_{k+1}\} \in z_{k+1}S_{i-1,k}} a_1 a_2 \cdots a_{i-1} z_{k+1}$$

$$= \sum_{\{a_1,a_2,...,a_i\} \in S_{i,k}} a_1 a_2 \cdots a_i + \sum_{\{a_1,a_2,...,a_{i-1},a_i\} \in z_{k+1}S_{i-1,k}} a_1 a_2 \cdots a_{i-1} a_i.$$

Since $\{a_1, a_2, ..., a_{i-1}, a_i\} \in z_{k+1}S_{i-1,k}$

$\Leftrightarrow$ there exists $j \in N_i$ such that $a_j = z_{k+1}$ and

$a_1, a_2, ..., a_{j-1}, a_{j+1}, ..., a_{i-1}, a_i \in C_k$

$\Leftrightarrow \{a_1, a_2, ..., a_{j-1}, z_{k+1}, a_{j+1}, ..., a_{i-1}, a_i\} \in R_{k+1}$

$\Leftrightarrow \{a_1, a_2, ..., a_{i-1}, a_i\} \in R_{i,k+1}$,

we obtain $z_{k+1}S_{i-1,k} = R_{k+1}$. Therefore

$$C_{i,k} + z_{k+1}C_{i-1,k} = \sum_{\{a_1,a_2,...,a_i\} \in S_{i,k}} a_1 a_2 \cdots a_i + \sum_{\{a_1,a_2,...,a_i\} \in R_{k+1}} a_1 a_2 \cdots a_i.$$

Since $S_{i,k} \cap R_{i,k+1} = \emptyset$ and $S_{i,k} \cup R_{i,k+1} = S_{i,k+1}$, we obtain

$$C_{i,k} + z_{k+1}C_{i-1,k} = \sum_{\{a_1,a_2,...,a_i\} \in S_{i,k} \cup R_{i,k+1}} a_1 a_2 \cdots a_i = \sum_{\{a_1,a_2,...,a_i\} \in S_{i,k+1}} a_1 a_2 \cdots a_i = C_{i,k+1}.$$

Hence $C_{i,k} + z_{k+1}C_{i-1,k} = C_{i,k+1}$.

By mathematical induction, this Lemma is true for all positive integer n. □

Lemma 3.2. Let n be a positive integer. If $z_1, z_2, ..., z_n$ are distinct complex numbers and $C_n = \{z_1, z_2, ..., z_n\}$, then

$$\prod_{i=1}^{n}(m-z_i)=\sum_{i=0}^{n-1}(-1)^i C_{i,n}m^{n-i}+(-1)^n C_{n,n}. \qquad (3.2)$$

Proof: We prove by mathematical induction. For $n=1,$ let z_1 be a complex number. Then since $C_{0,1}=1$ and $C_{1,1}=z_1,$

$$\prod_{i=1}^{1}(m-z_i)=m-z_1=(-1)^0 C_{0,1}m^{1-0}+(-1)^1 C_{1,1}=\sum_{i=0}^{0}(-1)^i C_{i,1}m^{1-i}+(-1)^1 C_{1,1}$$
$$=\sum_{i=0}^{n-1}(-1)^i C_{i,n}m^{n-i}+(-1)^n C_{n,n}.$$

This implies that (3.2) is true for $n=1.$

Next, we let k be a positive integer. Suppose that if $z_1,z_2,...,z_k$ are distinct complex numbers and $C_k=\{z_1,z_2,...,z_k\},$ then

$$\prod_{i=1}^{k}(m-z_i)=\sum_{i=0}^{n-1}(-1)^i C_{i,n}m^{k-i}+(-1)^k C_{k,k}.$$

Let $z_1,z_2,...,z_k,z_{k+1}$ be distinct complex numbers and $C_{k+1}=\{z_1,z_2,...,z_{k+1}\}.$ Since

$$\prod_{i=1}^{k+1}(m-z_i)=(m-z_{k+1})\prod_{i=1}^{k}(m-z_i),$$

by the inductive hypothesis, we obtain

$$\prod_{i=1}^{k}(m-z_i)=\sum_{i=0}^{k-1}(-1)^i C_{i,k}m^{k-i}+(-1)^k C_{k,k},$$

and thus

$$\prod_{i=1}^{k+1}(m-z_i)=(m-z_{k+1})\left(\sum_{i=0}^{k-1}(-1)^i C_{i,k}m^{k-i}+(-1)^k C_{k,k}\right)$$
$$=\sum_{i=0}^{k-1}(-1)^i C_{i,k}m^{k+1-i}-\sum_{i=0}^{k-1}(-1)^i C_{i,k}m^{k-i}z_{k+1}+(-1)^k C_{k,k}m-(-1)^k z_{k+1}C_{k,k}$$
$$=m^{k+1}+\sum_{i=1}^{k-1}(-1)^i C_{i,k}m^{k+1-i}-\sum_{i=0}^{k-2}(-1)^i C_{i,k}m^{k-i}z_{k+1}$$
$$+(-1)^k\left(z_{k+1}C_{k-1,k}+C_{k,k}\right)m+(-1)^{k+1}C_{k+1,k+1}.$$

By Lemma 3.1, we obtain $z_{k+1}C_{k-1,k}+C_{k,k}=C_{k,k+1}$ and therefore

$$\prod_{i=1}^{k+1}(m-z_i)=m^{k+1}+\sum_{i=1}^{k-1}(-1)^i C_{i,k}m^{k+1-i}-\sum_{i=0}^{k-2}(-1)^i C_{i,k}m^{k-i}z_{k+1}$$
$$+(-1)^k C_{k,k+1}m+(-1)^{k+1}C_{k+1,k+1}.$$

Let $j=i-1.$ Then $i=j+1$ and thus

$$\prod_{i=1}^{k+1}(m-z_i)=m^{k+1}+\sum_{j=0}^{k-2}(-1)^{j+1}C_{j+1,k}m^{k-j}-\sum_{i=0}^{k-2}(-1)^i C_{i,k}m^{k-i}z_{k+1}$$

$$+(-1)^k C_{k,k+1}m+(-1)^{k+1}C_{k+1,k+1}$$

$$=m^{k+1}+\sum_{i=0}^{k-2}(-1)^{i+1}C_{i+1,k}m^{k-i}+\sum_{i=0}^{k-2}(-1)^{i+1}z_{k+1}C_{i,k}m^{k-i}$$

$$+(-1)^k C_{k,k+1}m+(-1)^{k+1}C_{k+1,k+1}$$

$$=m^{k+1}+\sum_{i=0}^{k-2}(-1)^{i+1}C_{i+1,k+1}m^{k-i}+(-1)^k C_{k,k+1}m+(-1)^{k+1}C_{k+1,k+1}$$

by Lemma 3.1. Let $j=i+1$. Then $i=j-1$ and so

$$\prod_{i=1}^{k+1}(m-z_i)=m^{k+1}+\sum_{j=1}^{k-1}(-1)^j C_{j,k+1}m^{k+1-j}+(-1)^k C_{k,k+1}m+(-1)^{k+1}C_{k+1,k+1}$$

$$=m^{k+1}+\sum_{i=1}^{k-1}(-1)^i C_{j,k+1}m^{k+1-i}+(-1)^k C_{k,k+1}m+(-1)^{k+1}C_{k+1,k+1}$$

$$=\sum_{i=0}^{k}(-1)^i C_{i,k+1}m^{k+1-i}+(-1)^{k+1}C_{k+1,k+1} \qquad \square$$

The following corollary is a direct consequence of Lemma 3.2.

Corollary 3.1. Let n be a positive integer. If $z_1,z_2,...,z_n$ are distinct complex numbers and $C_n=\{z_1,z_2,...,z_n\}$, then $z_1,z_2,...,z_n$ are zeroes of the polynomial

$$\sum_{i=1}^{n-1}(-1)^i C_{i,n}m^{n-i}+(-1)^n C_{n,n}. \tag{3.3}$$

Lemma 3.3. Let n be a positive integer. If $z_1,z_2,...,z_n$ are distinct complex numbers and $C_n=\{z_1,z_2,...,z_n\}$, then

$$a_0=(-1)^n C_{n,n},\quad a_n=1 \text{ and } a_j=(-1)^{n-j}C_{n-j,n}+\sum_{i=1}^{n-j}(-1)^{i+1}N_{i,i+j-1}a_{i+j} \tag{3.4}$$

for every $j=1,2,...,n-1$ if and only if

$$a_n m^n+\sum_{j=1}^{n-1}m^{n-j}\sum_{i=0}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j}+a_0=\sum_{i=0}^{n-1}(-1)^i C_{i,n}m^{n-i}+(-1)^n C_{n,n}. \tag{3.5}$$

Proof: Suppose that (3.4) is true for all $j=1,2,...,n-1$. Then for any $j=1,2,...,n-1$,

$$a_j+\sum_{i=1}^{n-j}(-1)^i N_{i,i+j-1}a_{i+j}=(-1)^{n-j}C_{i,n},$$

that is for every $s=1,2,...,n-1$,

$$a_s+\sum_{i=1}^{n-s}(-1)^i N_{i,i+s-1}a_{i+s}=(-1)^{n-s}C_{i,n}.$$

Let $s=n-j$. Then $j=n-s$. and thus for every $j=1,2,...,n-1$,

$$a_{n-j}+\sum_{i=1}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j} = (-1)^j C_{i,n}.$$

Since $N_{0,n-j-1}=1$,

$$(-1)^0 N_{0,n-j-1}a_{n-j}+\sum_{i=1}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j} = (-1)^j C_{i,n}$$

and thus

$$\sum_{i=0}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j} = (-1)^j C_{i,n},$$

for every $j=1,2,...,n-1$. Multiplying both sides of this equation by m^{n-j} , we obtain

$$m^{n-j}\sum_{i=0}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j} = (-1)^j C_{i,n}m^{n-j}.$$

Adding this $n-1$ equations, we have

$$\sum_{j=1}^{n-1}m^{n-j}\sum_{i=0}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j} = \sum_{j=1}^{n-1}(-1)^j C_{i,n}m^{n-j}.$$

That is

$$a_n m^n+\sum_{j=1}^{n-1}m^{n-j}\sum_{i=0}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j}+a_0 = \sum_{i=0}^{n-1}(-1)^i C_{i,n}m^{n-i}+(-1)^n C_{n,n}.$$

Conversely, assume that the equation (3.5) is true. Since $1,m,m^2,...,m^n$ are linearly independent, by undetermined coefficients, we obtain

$$a_n=1,\quad a_0=(-1)^n C_{n,n}$$

and for every $j=1,2,...,n-1$,

$$\sum_{i=0}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j} = \sum_{j=1}^{n-1}(-1)^j C_{j,n},$$

that is

$$a_{n-j}+\sum_{i=1}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j} = (-1)^j C_{j,n},$$

Let $s=n-j$. Then for every $s=1,2,...,n-1$,

$$a_s+\sum_{i=1}^{n-s}(-1)^i N_{i,i+s-1}a_{i+s} = (-1)^{n-s} C_{n-s,n}$$

and thus for every $j=1,2,...,n-1$,

$$a_j = (-1)^{n-j}C_{n-j,n}+\sum_{i=1}^{n-j}(-1)^{i+1}N_{i,i+j-1}a_{i+j}.$$

The proof is complete. □

Corollary 3.2. Let k be a positive integer and $\alpha_1,\alpha_2,...,\alpha_k,\beta_1,\beta_2,...,\beta_k,a_0,a_1,...,a_{2k}$ be real numbers with $a_n \neq 0$. If $z_1,z_2,...,z_{2k}$ are distinct complex numbers and $C_{2k} = \{z_1,z_2,...,z_{2k}\}$ such that $z_1 = \alpha_1+\beta_1 i$, $z_2 = \alpha_2+\beta_2 i,...,z_k = \alpha_k+\beta_k i$, $z_{k+1}=\overline{z_1}$, $z_{k+2}=\overline{z_2},...,$ $z_{2k}=\overline{z_k}$. Then

$$a_0 = (-1)^{2k}C_{2k,2k},\quad a_{2k}=1 \text{ and } a_j = (-1)^{2k-j}C_{2k-j,2k} + \sum_{i=1}^{2k-j}(-1)^{i+1}N_{i,i+j-1}a_{i+j} \quad (3.6)$$

for every $j=1,2,...,n-1$ if and only if $z_1,z_2,...,z_{2k}$ are the zeroes of polynomial

$$a_{2k}m^{2k}+\sum_{j=1}^{2k-1}m^{2k-j}\sum_{i=0}^{j}(-1)^i N_{i,2k+i-j-1}a_{2k+i-j}+a_0. \quad (3.7)$$

Proof: By Lemma 3.3 with $n=2k$, the formulas (3.6) is true for every $j=1,2,...,2k-1$ if and only if

$$a_{2k}m^{2k}+\sum_{j=1}^{2k-1}m^{2k-j}\sum_{i=0}^{j}(-1)^i N_{i,2k+i-j-1}a_{2k+i-j}+a_0 = \sum_{i=0}^{2k-1}(-1)^i C_{i,2k}m^{2k-i}+(-1)^{2k}C_{2k,2k}.$$

By Corollary 3.1 with $n=2k$, we obtain

$$\prod_{i=1}^{2k}(m-z_i) = \sum_{i=0}^{2k-1}(-1)^i C_{i,2k}m^{2k-i}+(-1)^{2k}C_{2k,2k}.$$

and thus (3.6) is true for every $j=1,2,...,n-1$ if and only if

$$\prod_{i=1}^{n}(m-z_i) = a_n m^n+\sum_{j=1}^{n-1}m^{n-j}\sum_{i=0}^{j}(-1)^i N_{i,n+i-j-1}a_{n+i-j}+a_0.$$

Thus (3.6) holds for every $j=1,2,...,n-1$, if and only if $z_1,z_2,...,z_n$ are zeroes of the polynomial (3.7). □

The following main theorem is a direct consequence of Corollary 3.2 and Theorem 2.2 with $n=2k$.

Theorem 3.1. Let k be a positive integer and $\alpha_1,\alpha_2,...,\alpha_k,\beta_1,\beta_2,...,\beta_k,a_0,a_1,...,a_{2k}$ be real numbers with $a_{2k}\neq 0$. If $z_1,z_2,...,z_{2k}$ are distinct complex numbers and $C_{2k}=\{z_1,z_2,...,z_{2k}\}$ such that $z_1=\alpha_1+\beta_1 i$, $z_2=\alpha_2+\beta_2 i,...,z_k=\alpha_k+\beta_k i$, $z_{k+1}=\overline{z_1}$, $z_{k+2}=\overline{z_2},...,$ $z_{2k}=\overline{z_k}$. Then (3.6) is true for every $j=1,2,...,2k-1$ if and only if

$$y=\sum_{j=1}^{k}x^{\alpha_j}\left(c_{2j-1}\sin(\beta_j \ln x)+c_{2j}\cos(\beta_j \ln x)\right) \quad (3.8)$$

is the general solution of homogeneous Cauchy-Euler equation

$$a_{2k}x^{2k}\frac{d^{2k}y}{dx^{2k}}+a_{2k-1}x^{2k-1}\frac{d^{2k-1}y}{dx^{2k-1}}+\cdots+a_1 x\frac{dy}{dx}+a_0 y=0 \quad (3.9)$$

on an open interval $(0,\infty)$, where $c_1, c_2, ..., c_{2k}$ are arbitrary constants.

The application of Theorem 3.1 is to establish a Cauchy-Euler equation of order n together with its general solution on $(0,\infty)$, of the form (3.8).

4. Conclusion

We give every Cauchy-Euler differential equation from its general solution that depends only on a given finite numbers of distinct complex numbers. In the future, we will devote our attention to the family of all Cauchy-Euler differential equations that have general solutions depending only on a complex number.

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