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Size Multipartite Ramsey Numbers for Small Paths vs. $K_{2,n}$

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Abstract. Let G and H be finite graphs without loops and multiple edges. We use the notation $K_{j\times s}\rightarrow (G,H)$ to mean that if the edges of the complete graph $K_{j\times s}$ are coloured by the two colours red and blue, then either the red subgraph of $K_{j\times s}$ contains a copy of G, or the blue subgraph of $K_{j\times s}$ contains a blue copy of H. The size Ramsey multipartite number $m_j\,(P_3,K_{2,n})$ is defined as the smallest natural number s such that $K_{j\times s}\rightarrow (P_3,K_{2,n})$. In this paper, we obtain the exact values of the size Ramsey numbers $m_y(P_3,K_{2,n})$ and $m_y(P_4,K_{2,n})$ for $j \geq 3$.

Keywords: Ramsey theory, Multipartite Ramsey numbers

AMS Mathematics Subject Classification (2010): 05C55, 05D10

1. Introduction

Let G and H be two finite graphs without loops and multiple edges. Let the complete graph on n vertices and the complete balance multipartie graph having j multipartite sets of size s be denoted by K_n and $K_{j\times s}$ respectively. The write $K_{j\times s}\rightarrow (G,H)$ to mean that if edges of the complete graph $K_{j\times s}$ are coloured by the two colours red and blue, then there is a copy of G in red, or a copy of H in blue. In particular, the smallest positive integer n such that $K_n\rightarrow (G,H)$ is defined as the Ramsey number $r(G,H)$. Moreover, diagonal classical Ramsey number $r(n,n)$ is defined as $r(K_n,K_n)$. The exact determination of these diagonal classical Ramsey numbers have been studied for a few decades (see Radziszowski, 2017 for a survey) but sadly not much progress has been done even in the case of $r(5,5)$. In the last decade, using this idea of the original classical Ramsey numbers and of the size Ramsey numbers, the notion of size multipartite Ramsey numbers were introduced by Burger and Vuuren (i.e., Burger et al., 2004) by exploring the two colourings of multipartite graph $K_{j\times s}$ instead of the complete graph. More formally, *size Ramsey multipartite number* $m_j(G,H)$ is defined as the smallest natural number s such that $K_{j\times s}\rightarrow (G,H)$. Size multipartite Ramsey numbers have not been studied in detail up to now. Some of the known results are by Syafrizal, Baskaro and Uttunggadewa (i.e., Syfrizal et al., 2005).

Notation

Given a simple graph $G = (V,E)$, the *order* and the *size* of the graph are defined as $|V(G)|$ and $|E(G)|$ respectively. Given a vertex $v \in V(G)$ we define the neighborhood of v as the set of vertices adjacent to v in G and is denoted by $N(v)$. We also define $|N(v)|$ as the *degree* of such a vertex. A path on n vertices in a graph $G = (V,E)$ denoted by $P_n(V,E)$, is the subgraph consisting of the vertex set $\{a_1,a_2,...,a_n\}$ and the edge set $\{(a_1,a_2), (a_2,a_3), (a_3,a_4), ..., (a_{n-1},a_n)\}$ respectively.

2. Size Ramsey numbers related to paths of size three verses a certain class of complete bipartite graphs

Lemma 1. *Let $j \geq 3$ and $n \geq 1$. Then,*

$$\left\lceil \frac{n+1}{j-1} \right\rceil \leq m_j(P_3, K_{2,n}) \leq \left\lceil \frac{n+2}{j-1} \right\rceil.$$

Proof: $m_j(P_3,K_{2,n}) = 1$ when $n \leq j-2$. Therefore, if $\left\lceil \frac{n+1}{j-1} \right\rceil = 1$the result followstrivially. So,for the rest of the proof, we assume that $\left\lceil \frac{n+1}{j-1} \right\rceil \geq 2$.

First to find an upper bound, consider any red P_3-free red/blue colouring of $K_{j\times s}$, where$s = \left\lceil \frac{n+2}{j-1} \right\rceil$. Let $G = H_R \oplus H_B$ where H_R and H_B are the red and blue subgraph of Ginduced by the red and blue colouring, respectively. Since H_R has no P_3, H_R will contain two vertices, v_1 and v_2 belonging to the same partition A, such that each of these vertices will have at most red degree one. This will force a $K_{2,m}$ in H_B, where

$$m = (j-1)s - 2 = (j-1)\left\lceil \frac{n+2}{j-1} \right\rceil - 2 \geq n$$

with the highest degree vertices of $K_{2,m}$ chosen to be v_1 and v_2.Therefore,

$$m_j(P_3, K_{2,n}) \leq \left\lceil \frac{n+2}{j-1} \right\rceil$$

Next to find a lower bound, consider the colouring given by $K_{j\times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{n+1}{j-1} \right\rceil - 1$, such that H_R is a matching as illustrated in the following graphs corresponding to the two cases s even and s is odd. If s is odd and j is odd as indicated in the second figure, one vertex will be a isolated vertex in red. If s is odd and j is odd as indicated in the second figure, the red graph will consist of a perfect matching and the edge (x,y) will be coloured red.

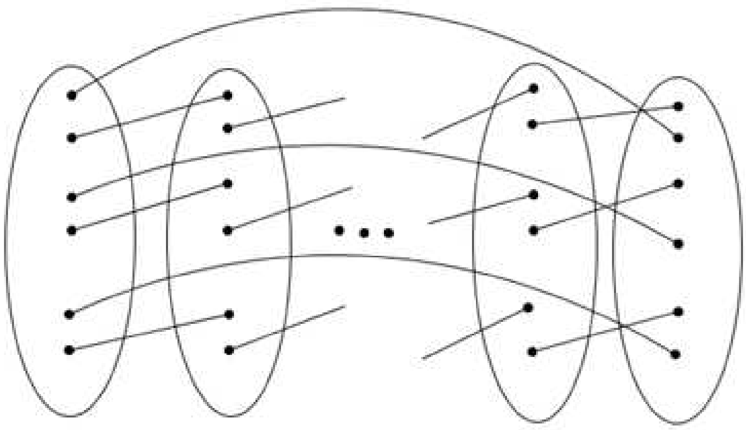

Figure 1: (a) If s is even

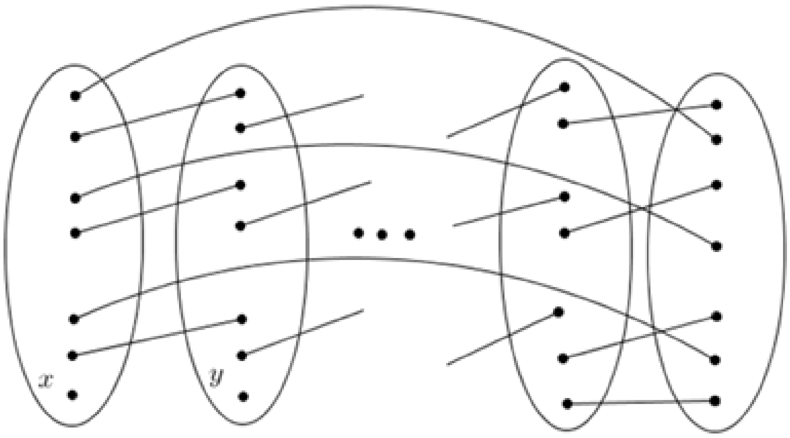

Figure 1: (b) If s is d

If j is even, (x,y) is a red edge. Otherwise x is an isolated vertex in H_R.
As seen above, in both these cases the graph has no red P_3. Also note that if $s \geq 2$, then, $s(j-2) \leq s(j-1) - 2$ if $s \geq 2$. Therefore, if $K_{2,m}$ is contained in the graph then

$$m \leq (j-1)s - 1 = \left(\left\lceil \frac{n+2}{j-1} \right\rceil - 1\right)(j-1) < \left(\frac{n+2}{j-1}\right)(j-1) - 1 = n \quad \text{if } s \geq 2$$

and

$$m \leq s(j-2) = (j-2) \leq n \text{ if } s = 1$$

Therefore, in both cases the graph contains no blue $K_{2,n}$.
Hence, $m_j(P_3, K_{2,n}) \geq \left\lceil \frac{n+1}{j-1} \right\rceil$.Hence the result.□

Theorem 2. *$m_j(P_3,K_{2,n}) = 1$ if $n \leq j-2$.*
Also, if $n > j-2$,

$$m_j(P_3, K_{2,n})$$

Proof: As seen in lemma 1, $m_j(P_3,K_{2,n}) = 1$ if $n \leq j-2$.
Hence, assume that $s = \left\lceil \frac{n+2}{j-1} \right\rceil \geq 2$. When, $n+1 \neq 0 \; mod(j-1)$, we know that $\left\lceil \frac{n+1}{j-1} \right\rceil = \left\lceil \frac{n+2}{j-1} \right\rceil$. Therefore, when $n+1 \neq 0 \; mod(j-1)$ the theorem directly follows from the lemma 1.

Hence, we may assume that $n+1 = 0 \; mod(j-1)$ and $s \geq 2$. Thus, we are left with only the following three cases.

Case 1: If $s = \left\lceil \frac{n+1}{j-1} \right\rceil$ is even.

Consider the colouring given by $K_{j \times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{n+2}{j-1} \right\rceil - 1$, such that H_R is a perfect matching as shown in the following diagram.

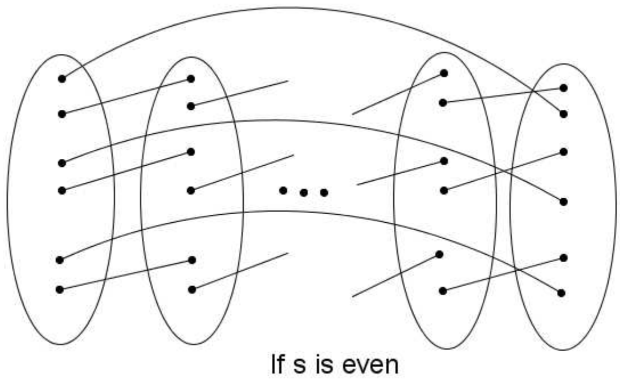

Figure 2: Case 1

Then, the graph has no red P_3. Moreover, if $K_{2,m}$ is contained in the graph then as $s(j-2) \leq s(j-1) - 2$ we get

$$m \leq (j-1)s - 2 = \left(\left\lceil\frac{n+2}{j-1}\right\rceil - 1\right)(j-1) - 2 < \left(\frac{n+2}{j-1}\right)(j-1) - 2 = n$$

Therefore, the graph contains no blue $K_{2,n}$. Hence, $m_j(P_3, K_{2,n}) \geq \left\lceil\frac{n+2}{j-1}\right\rceil$.

Therefore, by the lemma 1, in this case we get $m_j(P_3, K_{2,n}) = \left\lceil\frac{n+2}{j-1}\right\rceil$.

Case 2: If j is even.

Consider the colouring given by $K_{j\times s} = H_R \oplus H_B$, where $s = \left\lceil\frac{n+2}{j-1}\right\rceil - 1$, such that H_R is a matching as illustrated in the following diagram.

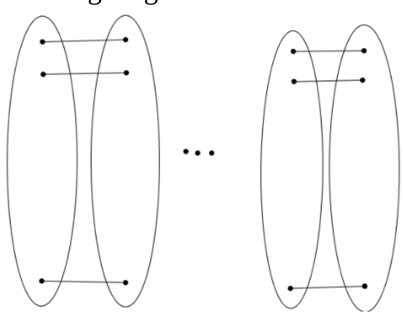

Figure 3: Case 2

Then, this graph has no red P_3. Moreover, if the corresponding blue graph has no $K_{2,n}$. Therefore, in this case, we get $m_j(P_3, K_{2,n}) \geq \left\lceil\frac{n+2}{j-1}\right\rceil$.

That is $m_j(P_3, K_{2,n}) = \left\lceil\frac{n+2}{j-1}\right\rceil$.

Case 3: If $\left\lceil\frac{n+1}{j-1}\right\rceil$ and j are odd.

Consider any red P_3-free red/blue colouring of $K_{j\times s}$, where $s = \left\lceil\frac{n+1}{j-1}\right\rceil$. Let $G = H_R \bigoplus H_B$ where H_R and H_B are the red and blue subgraph of G induced by the red and blue colouring, respectively. Since H_R has no P_3 and $j \times s$ is odd, H_R will contain one isolated vertex v_1. Let v_2 be another vertex of the same partition v_1 belongs to. This will force a $K_{2,m}$ in H_B, where

$$m = (j-1)s - 1 = \left(\left\lceil\frac{n+1}{j-1}\right\rceil - 1\right) \geq n$$

with the highest degree vertices of $K_{2,m}$ chosen to be v_1 and v_2. Therefore,

$$m_j(P_3, K_{2,n}) \leq \left\lceil\frac{n+1}{j-1}\right\rceil.$$

Hence in this case, $m_j(P_3, K_{2,n}) = \left\lceil\frac{n+1}{j-1}\right\rceil$ as required.

3. Size Ramsey numbers related to paths of size four verses certain class of complete bipartite graphs

The following definitions follow from a paper by author et al 2016.

Definition 3. (Bad colourings) A (red and blue) colouring of $K_{j\times s}(= H_R \bigoplus H_B)$ is called a bad colouring if the red connected components of HR consists of three cycles and at most two disjoint edges.

Definition 4. (Colouring of $K_{j\times s}$ generated by a $s\times t$ matrix) Let $A = (a_{im})_{sat}$ represent an matrix consisting of distinct elements in each column. Then $G = G(A)$ the multipartite graph with j partite sets generated by A, is defined by $V(G) = \{v_{k,i} | 1 \leq i \leq s, 1 \leq k \leq j\}$and $E(G) = \{(v_{k,i}, v_{k',i'}) |\ a_{ik} = a_{k',i'}\}$, where the j partite sets are respectively given by $V_k = \{v_{k,i} | i = 1,...,s\}$ for $k = 1,...,j$.

The red and blue colouring of $K_{j\times s}$ given by $K_{j\times s} = H_R \bigoplus H_B$ such that $H_R = G$ is said to be the two colouring generated by A.

Theorem 5. *Let $j \geq 6$.*

a) If $n < 3j - 7$ then $m_j(P_4, K_{2,n}) \in \{1,2,3\}$.

b) If $n \geq 3j - 7$ then,

$m_j(P_4, K_{2,n})$

Proof: (a) The above theorem is a direct consequence of the following two propositions.

Proposition 6. (a) $m_j(P_4, K_{2,n}) \leq \left\lceil\frac{n+4}{j-1}\right\rceil$.

(b) *If* $\left\lceil\frac{n+3}{j-1}\right\rceil$ j $\neq 0 \ mod\ 3$ then $m_j(P_4, K_{2,n}) \leq \left\lceil\frac{n+3}{j-1}\right\rceil$.

Proof: Before we start off with the proofs noting that, clearly (a) is true if $\left\lceil\frac{n+4}{j-1}\right\rceil \leq 1$ and (b) is true if $\left\lceil\frac{n+3}{j-1}\right\rceil \leq 1$ as $m_j(P_4, K_{2,n}) = 1$ if $n \leq j - 3$.

(a) To find an upper bound, consider any arbitrary red P_4-free red/blue colouring of $K_{j\times s}=H_R \oplus H_B$, where $s = \left\lceil\frac{n+4}{j-1}\right\rceil >1$

Claim 1a: All connected components of H_R consists of at most 3 vertices.
Assume the claim is false. Then in order to avoid a P_4, there will exist a vertex v that will be adjacent to at least three mono-valent vertices v_2,v_3 and v_4 in H_R. These vertices belong to distinct partite sets; as otherwise any two of these vertices belonging to the same partite set will be forced to be in some blue $K_{2,n}$ (this is true because

$$(j-1)s-1=(j-1)\left\lceil\frac{n+4}{j-1}\right\rceil-1\geq n),$$

a contradiction. Therefore, assume that the three three mono-valent vertices v_2,v_3 and v_4 in H_R belong to distinct partite sets A_1 ,A_2 and A_3 respectively. Consider the vertex v_2 and let u be any other vertex in the same partite set v_2 belongs to. Repeating the previous argument we will get that u will be adjacent to at least three mono-valent vertices u_2,u_3 and u_4 belong to distinct partite sets B_2,B_2 and B_3. Proceeding in this manner, we will arrive at a contradiction as the number of monovalent vertices of H_R is finite. Hence the claim.
Therefore, by the previous claim, given any pair of vertices v_1 and v_2 belonging to the same partitions A, we get that v_1 and v_2 are adjacent to at most four vertices in $H_R\backslash\{A\}$.
This forces a $K_{1,m}$ where

$$m\geq(j-1)s-4=(j-1)\left\lceil\frac{n+4}{j-1}\right\rceil-4\geq n$$

Thus, every colouring of $K_{j\times s}$ contains a red P_4 or a blue $K_{2,n}$. Hence

$$m_j(P_4,K_{2,n})\leq\left\lceil\frac{n+4}{j-1}\right\rceil.$$

(b) In this case to find an upper bound, consider any arbitrary red P_4-free red/blue colouring of $K_{j\times s}= H_R \oplus H_B$, where $s = \left\lceil\frac{n+3}{j-1}\right\rceil >1$.

Claim 1b: All connected components of H_R consists of at most 3 vertices.
Assume the claim is false. Then in order to avoid a P_4, there will exist a vertex v that will be adjacent to at least three mono-valent vertices v_2,v_3 and v_4 in H_R. These vertices belong to distinct partite sets; as otherwise any two of these vertices belonging to the same partite set will be in some blue $K_{2,n}$ (as $(j-1)s-1=(j-1)\left\lceil\frac{n+3}{j-1}\right\rceil-1\geq n$), acontradiction. Therefore, assume that the three three mono-valent vertices v_2,v_3 and v_4 in H_R belong to distinct partite sets A_1,A_2 and A_3 respectively. Consider the vertex v_2 and let u be any other vertex in the same partite set v_2 belongs to. Repeating the previous argument we will get u that will be adjacent to at least three mono-valent vertices u_2,u_3 and u_4 belong to distinct partite sets B_2,B_2 and B_3. Proceeding in this manner we will arrive at a contradiction as the number of monovalent vertices of H_R is finite. Hence the claim.

By the above claim all the connected components of H_R must be of size at most three. However, as sj= 0 mod3 we get that one of the components must be of size one or two. Therefore, there exists a vertex v of degree at most 1. Let v_1 be any other vertex belonging to the same partite set v belongs to. Then by the claim V_1 can be adjacent in red to at most two other vertices. we get that v and v_1 will be adjacent to at most 3vertices. But as, $K_{2,n}$

(as $(j-1)s-3=(j-1)\left\lceil\frac{n+3}{j-1}\right\rceil-3\geq n$, we get that v and v_1 will be the largest blue degree vertices of a blue $K_{1,n}$. Therefore, if $\left\lceil\frac{n+3}{j-1}\right\rceil$ j $\neq 0\ mod\ 3$ then $m_j(P_4,K_{2,n})\leq\left\lceil\frac{n+3}{j-1}\right\rceil$. □

Note that from the above proposition **(a)** part of the proposition follows as directly.

Proposition 7. *Suppose* $n\geq 3j-7$, (*i.e.* $s=(j-1)\left\lceil\frac{n+3}{j-1}\right\rceil-1\geq 2$). *Then we get* $m_j(P_4,K_{2,n})\geq\left\lceil\frac{n+4}{j-1}\right\rceil$. *Moreover, if* $j\neq 0$ *mod* 3 *and* $s\neq 2$ *mod* 3, *then we get* $m_j(P_4,K_{2,n})\geq\left\lceil\frac{n+3}{j-1}\right\rceil$.

Proof: The above theorem is a direct consequence of the results obtained in the following 4 cases.

Case 1: If $j=0$ *mod* 3.

Let $V_1,\ldots V_{3k-1},V_{3k}$ where $j=3k$ represent the j partite sets of $K_{j\times s'}$. Consider the colouring $K_{j\times s}=H_R\oplus H_B$, where $s'=\left\lceil\frac{n+4}{j-1}\right\rceil-1$, such that B_R is partitioned in to *ski* disjoint 3 cycles such that V_1,V_2,V_3 consists of s disjoint triangles, V_2,V_3,V_4 consists of s disjoint triangles and likewise continuing in this manner V_{3k-1},V_{3k-1},V_{3k} consists of s disjoint triangles.

Then, the graph has no red P_4. Moreover, if $K_{2,m}$ is a contained in H_B then

$$m\leq(j-1)s-4=\left(\left\lceil\frac{n+4}{j-1}\right\rceil-1\right)(j-1)-4<\left(\frac{n+4}{j-1}\right)(j-1)-4=n$$

Therefore, the graph contains no blue $K_{2,n}$. Hence, $m_j(P_4,K_{2,n})\geq\left\lceil\frac{n+4}{j-1}\right\rceil$.

Case 2: If $s=2$ *mod* 3.

Case 2.1: If $s=2$ *mod* 3 and $n+3=0$ *mod* $(j-1)$.

Then $\left\lceil\frac{n+3}{j-1}\right\rceil+1=\left\lceil\frac{n+4}{j-1}\right\rceil$. Consider the colouring generated on $K_{j\times s'}$, where $s'=\left\lceil\frac{n+4}{j-1}\right\rceil+1=3q$ where $s'=3q$, by the matrix $(A)_{3q\times j}$ given below. Note that in this colouring, all the vertices of B_R are partitioned in to 3 cycles,

$$\begin{pmatrix}
a_1 & b_1 & 3 & 3 & 3 & \ldots & j-2 & j-2 & j-2\\
b_1 & 2 & 2 & 2 & 5 & \ldots & j-3 & j-3 & a_1\\
1 & 1 & 1 & 4 & 4 & \ldots & j-4 & a_1 & b_1\\
a_2 & b_2 & j+1 & j+1 & j+1 & \ldots & 2j-4 & 2j-4 & 2j-4\\
b_2 & j & j & j & j+3 & \ldots & 2j-5 & 2j-5 & a_2\\
j-1 & j-1 & j-1 & j+2 & j+2 & \ldots & 2j-6 & a_2 & b_2\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
a_q & b_q & p-(j-5) & & & \ldots & p & p & p\\
b_q & p-(j-4) & & & & \ldots & p-1 & p-1 & a_q\\
p-(j-3) & & & & & \ldots & p-2 & a_q & b_q
\end{pmatrix}$$

where $p = q(j-2)$ and a_i's and b_i's distinct and consists of arbitrary assigned $2q$ elements of $\{p+1, p+2, \ldots, jq\}$. If $K_{2,m}$ is the largest blue $K_{2,m}$. Then as $s' \geq 3$

$$m \leq (j-1)s' - 4 = \left(\left\lceil \frac{n+4}{j-1} \right\rceil - 1\right)(j-1) - 4 < n$$

Therefore, the graph contains no blue $K_{2,n}$. Hence $m_j(P_4, K_{2,n}) \geq \left\lceil \frac{n+4}{j-1} \right\rceil$.

Subcase 2.2: $s \equiv 2 \mod 3$, $j = 1 \mod 3$ and $n + 3 \neq 0 \mod j - 1$.

As defined in subcase 2.1, let $A = (a_{i,j})_{3q \times j}$, where $s = 3q+2$, $r = \left\lfloor \frac{j}{3} \right\rfloor$ and $p = q(j-2)$.Consider the colouring generated on $K_{j \times s}$, where $s = \left\lceil \frac{n+3}{j-1} \right\rceil - 1$ by the matrix $B = (b_{i,j})_{(3q+2) \times j}$ given below where $p' = \max\{a_{i,j}\} + 2$. Note that in this colouring, all the vertices of B_R are partitioned in to 3 cycles except when $v_{1,s-1}$ and $v_{4,s}$ incident to the edge $(v_{1,s-1}, v_{4,s})$ corresponding to the $p' - 1$ valued double entry of the matrix.

$$\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,j-2} & a_{1,j-1} & a_{1,j} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,j-2} & a_{2,j-2} & a_{2,j} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{s-2,1} & a_{s-2,2} & a_{s-2,3} & a_{s-2,4} & \cdots & a_{s-2,j-2} & a_{s-2,j-s-1} & a_{s-2,j} \\
p'-1 & p' & p' & p' & \ldots & p'+r-1 & p'+r-1 & p'+r-1 \\
p'+r & p'+r & p'+r & p'-1 & \ldots & p'+2r-1 & p'+2r-1 & p'+2r-1
\end{pmatrix}$$

If m is the largest value such that $K_{2,m}$ is in R_B. Then as $j \geq 6$,

$$m \leq (j-1)s - 3 = \left(\left\lceil \frac{n+3}{j-1} \right\rceil - 1\right)(j-1) - 3 < n.$$

Therefore, we get that the graph contains no blue $K_{2,m}$.But as $n+3 \neq 0 \mod (j-1)$ we have $\left\lceil \frac{n+3}{j-1} \right\rceil = \left\lceil \frac{n+4}{j-1} \right\rceil$. Therefore, $m_j(P_4, K_{2,n}) \geq \left\lceil \frac{n+4}{j-1} \right\rceil$.

Subcase 2.3: $s = 2 \mod 3$, $j = 2 \mod 3$ and $n + 3 \neq 0 \mod (j-1)$.

As defined in subcase 2.1, let $A = (a_{i,j})_{3q \times j}$, where $s = 3q+2$, $r = \left\lceil \frac{j}{3} \right\rceil$ and $p = q(j-2)$. Consider the colouring generated on $K_{j \times s}$, where $s = \left\lceil \frac{n+3}{j-1} \right\rceil - 1$ by the matrix $D = (d_{i,j})_{(3q+2) \times j}$ given below with $p' = \max\{a_{i,j}\} + 2$. Note that in this colouring, all the vertices of B_R are partitioned in to 3 cycles except when $v_{1,s-1}$ and $v_{4,s-1}$ incident to the edge $(v_{1,s-1}, v_{4,s-1})$ corresponding to the $p' - 1$ valued double entry (i.e. $p' - 1$ appears exactly twice in the matrix) of the matrix or when $v_{2,s}$ and $v_{5,s}$ incident to the edge $(v_{2,s}, v_{5,s})$ corresponding to the $p' + 2r$ valued double entry of the matrix.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,j-2} & a_{1,j-1} & a_{1,j} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,j-2} & a_{2,j-2} & a_{2,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{s-2,1} & a_{s-2,2} & a_{s-2,3} & a_{s-2,4} & \cdots & a_{s-2,j-2} & a_{s-2,j-s-1} & a_{s-2,j} \\ p'-1 & p' & p' & p' & \ldots & p'+r-1 & p'+r-1 & p'+r-1 \\ p'+r & p'+2r & p'+r & p'+r & \ldots & p'+2r-1 & p'+2r-1 & p'+2r-1 \end{pmatrix}$$

If m is the largest value of $K_{2,m}$ in R_B. Then,

$$m \leq (j-1)s - 3 = \left(\left\lceil\frac{n+3}{j-1}\right\rceil - 1\right)(j-1) - 3 < n.$$

Therefore, we get that the graph contains no blue $K_{2,m}$. But as $n+3 \neq 0$ *mod* ($j-1$) we have $\left\lceil\frac{n+3}{j-1}\right\rceil = \left\lceil\frac{n+4}{j-1}\right\rceil$. Therefore, $m_j(P_4, K_{2,n}) \geq \left\lceil\frac{n+4}{j-1}\right\rceil$.

Case 3: If $s = 1$ *mod* 3.

Subcase 3.1: If $s = 1$ *mod* 3 and $j \equiv 1$ *mod* 3.

As defined in subcase 2.1, let $A = (a_{im})_{3q\times j}$, where $s = 3q+1$, $r = \left\lceil\frac{j}{3}\right\rceil$ and $p = q(j-2)$. Consider the colouring generated on $K_{j\times s}$, where $s = \left\lceil\frac{n+3}{j-1}\right\rceil - 1$ by the matrix $B = (bam)(3q+1)\times j$ given below where $p' = \max\{aim\} + 2$. Note that in this colouring, all the vertices of B_R are partitioned in to 3 cycles except when $v_{1,s}$ and $v_{4,s}$ incident to the edge $(v_{1,s}, v_{4,s})$ corresponding to the $p'-1$ valued double entry (i.e. $p'-1$ appears exactly twice in the matrix) of the matrix or when $v_{2,s}$ and $v_{5,s}$ incident to the edge $(v_{2,s}, v_{5,s})$ corresponding to the p'valued double entry of the matrix.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,j-2} & a_{1,j-1} & a_{1,j} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,j-2} & a_{2,j-2} & a_{2,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{s-1,1} & a_{s-1,2} & a_{s-1,s-1} & a_{s-1,4} & \cdots & a_{s-1,j-2} & a_{s-1,j-s-1} & a_{s-1,j} \\ p'-1 & p' & p'+1 & p'+1 & \ldots & p'+r & p'+r & p'+r \end{pmatrix}$$

If m is the largest value of $K_{2,m}$ in R_B. Then,

$$m \leq (j-1)s - 3 = \left(\left\lceil\frac{n+3}{j-1}\right\rceil - 1\right)(j-1) - 3 < n.$$

Therefore, we get that the graph contains no blue $K_{2,m}$. Hence, $m_j(P_4, K_{2,n}) \geq \left\lceil\frac{n+3}{j-1}\right\rceil$.

Subcase 3.2: $s = 1$ *mod* 3 and $j = 2$ *mod* 3.

As defined in case 1, let $A = (a_{i,j})_{3q\times j}$, where $s = 3q+1$, $r = \left\lceil\frac{j}{3}\right\rceil$ and $p = q(j-2)$. Consider the colouring generated on $K_{j\times s}$, where $s = \left\lceil\frac{n+3}{j-1}\right\rceil - 1$by the matrix $D = (d_{i,j})_{(3q+2)\times j}$ given below with $p' = \max\{a_{i,j}\} + 2$. Note that in this colouring, all the vertices of B_R are partitioned in to 3 cycles except when $v_{1,s}$ and $v_{4,s}$ incident to the edge $(v_{1,s}, v_{4,s})$ corresponding to the $p'-1$ valued double entry (i.e. $p'-1$ value appears exactly twice) of the matrix.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,j-2} & a_{1,j-1} & a_{1,j} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,j-2} & a_{2,j-2} & a_{2,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{s-1,1} & a_{s-1,2} & a_{s-1,3} & a_{s-1,4} & \cdots & a_{s-1,j-2} & a_{s-1,j-1} & a_{s-1,j} \\ p'-1 & p' & p' & p' & \ldots & p'+r-1 & p'+r-1 & p'+r-1 \end{pmatrix}$$

If m is the largest value of $K_{2,m}$ in R_B. Then as $j \geq 6$,

$$m \leq (j-1)s - 3 = \left(\left\lceil\frac{n+3}{j-1}\right\rceil - 1\right)(j-1) - 3 < n.$$

Therefore, we get that the graph contains no blue $K_{2,m}$.Hence, $m_j(P_4, K_{2,n}) \geq \left\lceil\frac{n+3}{j-1}\right\rceil$.

Case 4: If $s = 0$ *mod* 3.

Consider the colouring generated on $K_{j\times s}$ 0, where s= $\left\lceil\frac{n+3}{j-1}\right\rceil - 1 = 3q$ where $s' = 3q$, by the matrix $(A)_{3q\times j}$ given below.

Note that in this colouring, all the vertices of B_R are partitioned in to 3 cycles.

$$\begin{pmatrix} a_1 & b_1 & 3 & 3 & 3 & \ldots & j-2 & j-2 & j-2 \\ b_1 & 2 & 2 & 2 & 5 & \ldots & j-3 & j-3 & a_1 \\ 1 & 1 & 1 & 4 & 4 & \ldots & j-4 & a_1 & b_1 \\ a_2 & b_2 & j+1 & j+1 & j+1 & \ldots & 2j-4 & 2j-4 & 2j-4 \\ b_2 & j & j & j & j+3 & \ldots & 2j-5 & 2j-5 & a_2 \\ j-1 & j-1 & j-1 & j+2 & j+2 & \ldots & 2j-6 & a_2 & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_q & b_q & p-(j-5) & & & \ldots & p & p & p \\ b_q & p-(j-4) & & & & \ldots & p-1 & p-1 & a_q \\ p-(j-3) & & & & & \ldots & p-2 & a_q & b_q \end{pmatrix}$$

where $p = q(j-2)$ and a_i's and b_i's distinct and consists of arbitrary assigned $2q$ elements of $\{p+1, p+2, \ldots, jq\}$. If $K_{2,m}$ is the largest blue $K_{2,m}$. Then as $j \geq 6$,

$$m \leq (j-1)s - 4 = \left(\left\lceil\frac{n+3}{j-1}\right\rceil - 1\right)(j-1) - 4 < n-1.$$

Therefore, the graph contains no blue $K_{2,n}$. Hence, $m_j(P_4, K_{2,n}) \geq \left\lceil\frac{n+3}{j-1}\right\rceil$.□

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