Annals of Pure and Applied Mathematics
Vol. 19, No.1, 2019, 19-20
ISSN: 2279-087X (P), 2279-0888(online)
Published on 23 January 2019
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.596v19n1a3

Annals of
Pure and Applied Mathematics

A Note on the Diophantine Equation $p^x + (p + 1)^y = z^2$

Nechemia Burshtein

117 Arlozorov Street, Tel – Aviv 6209814, Israel
Email: anb17@netvision.net.il

Received 15 January 2019; accepted 22 January 2019

Abstract. Suvarnamani [3] proved that the equation $p^x + (p + 1)^y = z^2$ has the unique solution $(p, x, y, z) = (3, 1, 0, 2)$ when p is an odd prime and x, y, z are non-negative integers. In this note, we refute the uniqueness of this solution by presenting a counter-solution. When $p = 2$ is added to the list of odd primes, two solutions of the equation are demonstrated. Finally, when the prime p is substituted by a composite C, then from [4] it follows that the equation has exactly two such solutions which are exhibited.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

In [3] Suvarnamani proved that the equation

$$p^x + (p + 1)^y = z^2 \qquad (1)$$

in which p is an odd prime and x, y, z are non-negative integers, has the unique solution

$$(p, x, y, z) = (3, 1, 0, 2). \qquad (2)$$

In this note, we disprove the uniqueness of solution (2). We also extend the odd primes p to include the prime $p = 2$, and exhibit solutions with $p = 2$ which satisfy equation (1). Moreover, when p is replaced by a composite C, solutions of equation (1) are presented in which x, y, z are positive integers.

Let p be an odd prime. Then when $p = 3$ we have

Solution 1. $$3^2 + 4^2 = 5^2 = z^2$$

in which x, y, z are positive integers satisfying equation (1). **Solution 1** is known as the most famous "Pythagorean triple" which is also the smallest such triple and the trivial one.

For $p = 2$, we have

Solution 2. $$2^0 + 3^1 = 2^2 = z^2,$$

and x, y, z are non-negative integers which satisfy equation (1).

Furthermore, for $p = 2$, the following solution

Solution 3. $$2^4 + 3^2 = 5^2 = z^2$$

satisfies equation (1) with positive integers x, y, z.

A "Pythagorean triple" (abbreviated triple) is denoted (a, b, c) if a, b, c are positive integers satisfying $a^2 + b^2 = c^2$. Assume in (1) that p is replaced by a composite C. All values in [4] have been examined for $a = p$, $b = p + 1$ and for $a = C$, $b = C + 1$. In the first case, no solutions exist besides **Solution 1**. In the second case, exactly two solutions of equation (1) exist with positive integers x, y, z. These are:

Solution 4. $$20^2 + 21^2 = 29^2 = z^2,$$

Solution 5. $$119^2 + 120^2 = 169^2 = z^2.$$

Final remark. Whether other solutions to equation (1) exist or not remains an open question.

REFERENCES

1. N. Burshtein, On solutions to the diophantine equation $p^2 + q^2 = z^4$, *Annals of Pure and Applied Mathematics*, 18 (2) (2018) 135 – 138.
2. N. Burshtein, Solutions of the diophantine equation $p^x + (p + 6)^y = z^2$ when p, $(p + 6)$ are primes and $x + y = 2, 3, 4$, *Annals of Pure and Applied Mathematics*, 17 (1) (2018) 101 – 106.
3. A. Suvarnamani, On the diophantine equation $p^x + (p + 1)^y = z^2$, *International Journal of Pure and Applied Mathematics*, 94 (5) (2014) 689 – 692.
4. Integer Lists: "Pythagorean triples up to 2100" – TSM Resources.