Annals of Pure and Applied Mathematics
Vol. 19, No. 1, 2019, 53-57
ISSN: 2279-087X (P), 2279-0888(online)
Published on 12 February 2013
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.599v19n1a6

Annals of
Pure and Applied Mathematics

All the Solutions of the Diophantine Equations $(p+1)^x - p^y = z^2$ and $p^y - (p+1)^x = z^2$ when p is Prime and $x + y = 2, 3, 4$

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Received 5 February 2019; accepted 12 February 2019

Abstract. In this article we consider the two equations $(p+1)^x - p^y = z^2$ and $p^y - (p+1)^x = z^2$ in which $p \ge 2$ is prime, and x, y, z are positive integers. When $x + y = 2, 3, 4$, we establish that:

(i) The equation $(p+1)^1 - p^1 = z^2$ has a unique solution $(p, x, y, z) = (p, 1, 1, 1)$ for each and every prime $p \ge 2$.
(ii) The equation $p^2 - (p+1)^1 = z^2$ has the unique solution $(p, x, y, z) = (2, 1, 2, 1)$.
(iii) The equation $(p+1)^3 - p^1 = z^2$ has the unique solution $(p, x, y, z) = (2, 3, 1, 5)$.
(iv) No solutions exist for all other possible equations.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are [1, 2, 4] and others.

In this paper, we consider the two equations

$$(p+1)^x - p^y = z^2,$$
$$p^y - (p+1)^x = z^2$$

in which p is prime, x, y, z are positive integers, and $x + y = 2, 3, 4$.

In Section 2 the two equations are investigated for solutions. It is shown that the first equation has a unique trivial solution for each prime $p \ge 2$ when $x + y = 2$, and also a unique solution when

$p = 2$ and $x + y = 4$. The second equation has a unique solution when $p = 2$ and $x + y = 3$. For all other possibilities, the two equations have no solutions.

2. Solutions of $(p + 1)^x - p^y = z^2$ and $p^y - (p + 1)^x = z^2$

We note that when $x + y = 2, 3, 4$, each equation clearly has a total of six possibilities. In Theorem 2.1, all the twelve cases are considered. Besides primes $p \geq 2$, and positive integers x, y, z, all other values introduced in our discussion are positive integers.

Theorem 2.1. Suppose that in $(p + 1)^x - p^y = z^2$ and in $p^y - (p + 1)^x = z^2$, $p \geq 2$ is prime, and x, y, z are positive integers. If x, y satisfy $x + y = 2, 3, 4$, then:

(i) The equation $(p + 1)^1 - p^1 = z^2$ has a unique solution $(p, x, y, z) = (p, 1, 1, 1)$ for each and every prime $p \geq 2$.

(ii) The equation $p^2 - (p + 1)^1 = z^2$ has the unique solution when $p = 2$, $(p, x, y, z) = (2, 1, 2, 1)$.

(iii) The equation $(p + 1)^3 - p^1 = z^2$ has the unique solution when $p = 2$, $(p, x, y, z) = (2, 3, 1, 5)$.

(iv) No solutions exist for all other possible nine equations.

Proof: The twelve possible equations are considered separately, each of which is self-contained.

The case $x + y = 2$.

For $x + y = 2$, the only possibility is $x = 1$ and $y = 1$.

Case 1. $(p + 1)^1 - p^1 = z^2$.

For all primes $p \geq 2$ we have $z = 1$. The equation has infinitely many solutions of the form

$$(p, x, y, z) = (p, 1, 1, 1).$$

Case 2. $p^1 - (p + 1)^1 = z^2$.

We obtain $z^2 = -1$ which is impossible.

The equation $p^1 - (p + 1)^1 = z^2$ has no solutions.

This concludes the case $x + y = 2$.

The case $x + y = 3$.

For $x + y = 3$, we have $x = 1$ with $y = 2$ and $x = 2$ with $y = 1$.

Case 3. $(p + 1)^1 - p^2 = z^2$.

For each and every prime $p \geq 2$, the value $(p + 1)^1 - p^2 < 0$ yields $z^2 < 0$ which is impossible.

The equation $(p + 1)^1 - p^2 = z^2$ has no solutions.

Case 4. $p^2 - (p + 1)^1 = z^2$.

The equation yields $p^2 - z^2 = p + 1$ or $(p - z)(p + z) = p + 1$ where $p - z > 0$. If $z > 1$, then the equality is clearly impossible. When $z = 1$, then $p = 2$. The unique solution of this case is

Solution 1. $(p, x, y, z) = (2, 1, 2, 1)$.

Case 5. $(p + 1)^2 - p^1 = z^2$.
The equation $(p + 1)^2 - p^1 = z^2$ yields $p^2 + p + 1 = z^2$ or $p^2 + p = z^2 - 1$ and

$$p(p + 1) = (z - 1)(z + 1). \tag{1}$$

If $p = 2$, then $z^2 = 7$ is impossible. Hence, if the equation has a solution, then $p > 2$. From (1) it now follows that either $p \mid (z - 1)$ or $p \mid (z+ 1)$.

If $p \mid (z - 1)$ denote $Ap = z - 1$, and $Ap + 2 = z + 1$. From (1) we then obtain $p(p + 1) = (Ap)(Ap + 2)$ which is impossible for all values A. Thus $p \nmid (z - 1)$.

If $p \mid (z+ 1)$ denote $pB = z +1$, and $pB - 2 = z - 1$. Then (1) yields $p(p + 1) = (pB - 2)(pB)$ or $p + 1 = B(pB - 2)$, and B=1 is impossible. Since $p = (2B + 1)/(B^2 - 1)$ is never an integer for all $B >1$, therefore $p \nmid (z+ 1)$. Thus $p > 2$ is impossible, and case 5 is complete.

The equation $(p + 1)^2 - p^1 = z^2$ has no solutions.

Case 6. $p^1 - (p + 1)^2 = z^2$.
For each and every prime $p \geq 2$, the value $p^1 - (p + 1)^2 < 0$ and $z^2 < 0$ which is impossible.

The equation $p^1 - (p + 1)^2 = z^2$ has no solutions.

The case $x + y = 3$ is complete.

The case $x + y = 4$.

The case $x + y = 4$ has six possibilities demonstrated in the following cases 7 – 12.

Case 7. $(p + 1)^1 - p^3 = z^2$.
For each and every prime $p \geq 2$, the value $(p + 1)^1 - p^3 < 0$ implies $z^2 < 0$ which is impossible.

The equation $(p + 1)^1 - p^3 = z^2$ has no solutions.

Case 8. $p^3 - (p + 1)^1 = z^2$.
If $p = 2$, then $z^2 = 5$ which is impossible. Hence, if the equation has a solution, then $p > 2$.

The equation $p^3 - (p + 1)^1 = z^2$ yields $p^3 - p = z^2 + 1$ where $p(p^2 - 1) = z^2 + 1$ or

$$(p - 1)p(p + 1) = z^2 + 1, \qquad z \text{ is odd}. \tag{2}$$

Denote $z = 2T +1$. Thus,

$$z^2 + 1 = (2T + 1)^2 + 1 = 4T^2 + 4T + 2 = 2(2T^2 + 2T + 1), \tag{3}$$

and $z^2 + 1$ is a multiple of 2.
Since $p > 2$, it follows that $p - 1$ is even and $p + 1$ is even. Therefore, the left side of (2) is a multiple of at least 4. Then (2) and (3) yield

$$(p - 1)p(p + 1) = 2(2T^2 + 2T + 1)$$

a contradiction, implying that (2) does not exist.

The equation $p^3 - (p + 1)^1 = z^2$ has no solutions.

Case 9. $(p + 1)^2 - p^2 = z^2$.

The equation $(p + 1)^2 - p^2 = z^2$ implies $2p + 1 = z^2$ or

$$2p = z^2 - 1 = (z - 1)(z + 1) \qquad z \text{ is odd.} \tag{4}$$

If $p = 2$, then $z^2 = 5$ which is impossible. Hence $p > 2$. Since z is odd, it follows that each of the values $z - 1$ and $z + 1$ is even. Therefore, the right side of (4) is a multiple of at least 4, whereas the left side of (4) is a multiple of 2 since p is odd. Hence (4) does not exist.

The equation $(p + 1)^2 - p^2 = z^2$ has no solutions.

Case 10. $p^2 - (p + 1)^2 = z^2$.

For each and every prime $p \geq 2$, the value $p^2 - (p + 1)^2 < 0$ and $z^2 < 0$ which is impossible.

The equation $p^2 - (p + 1)^2 = z^2$ has no solutions.

Case 11. $(p + 1)^3 - p^1 = z^2$.

When $p = 2$, then $z = 5$, and we have

Solution 2. $(p, x, y, z) = (2, 3, 1, 5)$.

Suppose now that $p > 2$. The value z is odd, and denote $z = 2T + 1$. The equation $(p + 1)^3 - p = z^2$ yields $p^3 + 3p^2 + 2p + 1 = z^2$ or $p(p^2 + 3p + 2) = z^2 - 1$ and

$$p(p + 1)(p + 2) = (z - 1)(z + 1). \tag{5}$$

From (5) it follows that either $p \mid (z - 1)$ or $p \mid (z + 1)$.

If $p \mid (z - 1)$ denote $Ap = z - 1$, and $Ap + 2 = z + 1$. From (5) we then obtain $p(p + 1)(p + 2) = (Ap)(Ap + 2)$ which is impossible when $A = 1$. Therefore $A > 1$, and we have

$$(p + 1)(p + 2) = A(Ap+ 2). \tag{6}$$

The value $Ap + 2 = 1(p + 2) + (A - 1)p$, and from (6) $(p + 1)(p + 2) = A(1(p + 2) + (A - 1)p)$ or

$$(p + 2)((p + 1) - A) = A(A - 1)p. \tag{7}$$

Since $p > 2$ is prime, it follows from (7) that $p \mid ((p + 1) - A)$. But, $A > 1$ implies that this is impossible. Hence $p \nmid (z - 1)$.

If $p \mid (z + 1)$ denote $Bp = z + 1$, and $Bp - 2 = z - 1$. From (5) we have

$$p(p + 1)(p + 2) = (Bp - 2)(Bp) \tag{8}$$

which is impossible when $B = 1$. Hence $B > 1$. The value $Bp - 2 = -1(p + 2) + (B + 1)p$, and from (8) $(p + 1)(p + 2) = B(-1(p + 2) + (B + 1)p)$ or

$$(p + 2)((p + 1) + B) = B(B + 1)p. \tag{9}$$

Since $p > 2$ is prime, it follows from (9) that $p \mid (B + 1)$. Denote $Cp = B + 1$ and $B = Cp - 1$. Then from (8), we have

$$(p + 1)(p + 2) = ((Cp - 1)p - 2)(Cp - 1) = p(Cp - 1)^2 - 2(Cp - 1)$$

or

$$p^2 + 3p + 2 = p(p^2C^2 - 2Cp + 1) - 2Cp + 2$$

and

$$p(p + 3) = p(p^2C^2 - 2Cp + 1 - 2C).$$

Thus, after simplification by p

$$p(pC^2 - 2C - 1) = 3 - 1 + 2C = 2(C + 1). \qquad (10)$$

When $C = 1$, (10) yields $p(p - 3) = 4$ which is impossible, and hence $C > 1$. We now show for all primes $p > 2$ and all values $C > 1$ that equality (10) does not exist.

Since $p > 2$, it therefore suffices to show in (10) that if $pC^2 - 2C - 1 > C + 1$ for all $C > 1$, then (10) does not exist. The inequality $pC^2 - 2C - 1 > C + 1$ yields

$$p > (3C + 2) / C^2. \qquad (11)$$

For all values $C > 1$, it is easily seen that $((3C + 2) / C^2) \le 2 < p$ is indeed true. The validity of (11) is established, and hence (10) does not exist.

This completes Case 11.

The equation $(p + 1)^3 - p^1 = z^2$ has a unique solution when $p = 2$, namely **Solution 2**, and no solutions for all primes $p > 2$.

Case 12. $p^1 - (p + 1)^3 = z^2$.

For each and every prime $p \ge 2$, the value $p - (p + 1)^3 < 0$. Hence $z^2 < 0$ which is impossible.

The equation $p^1 - (p + 1)^3 = z^2$ has no solutions.

The proof of Theorem 2.1. is complete. □

Final remark. When $x + y = 2$ and $x + y = 4$, then $(p + 1)^x - p^y = z^2$ has a unique solution when $p = 2$, whereas $p^y - (p + 1)^x = z^2$ with $x + y = 3$ also has a unique solution when $p = 2$. Thus, each of the three possibilities $x + y = 2, 3, 4$ yields a solution when $p = 2$. Moreover, for $(p + 1)^x - p^y = z^2$ with $p = 2$, the following solution exists, namely

Solution 3. $3^4 - 2^5 = 7^2$, $x + y = 9$.

One may then ask whether there exist other solutions to $(p + 1)^x - p^y = z^2$ with $p = 2$ and $x + y > 4$. The question is also valid for $p^y - (p + 1)^x = z^2$. The same questions may be asked for both equations also when $p > 2$ and $x + y > 4$.

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