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On Solutions to the Diophantine Equations $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$ with Positive Integers x, y, z

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Abstract. In this paper, we consider the two equations $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$ in which x, y, z are positive integers. For $5^x + 103^y = z^2$, it is established that the equation has no solutions. For $5^x + 11^y = z^2$, when y is even it is shown that the equation has no solutions. For all values $1 \leq x \leq 14$ together with all odd values $1 \leq y \leq 9$, and up to $5^{14} + 11^9 = 8461463316$, it is determined that the equation has exactly three solutions which are exhibited.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this paper, we consider the two equations

$$5^x + 103^y = z^2,$$
$$5^x + 11^y = z^2$$

where x, y, z are positive integers. One could cite here many articles on the equation $p^x + q^y = z^2$. We provide here only a small number of related equations which include the prime 5 in particular, such as [1, 2, 4, 5, 6].

All other values introduced in this paper are also positive integers.

2. The equation $5^x + 103^y = z^2$ when x, y, z are positive integers

In the following theorem it is shown that $5^x + 103^y = z^2$ has no solutions when x, y, z are positive integers.

Theorem 2.1. The equation $5^x + 103^y = z^2$ has no solutions in positive integers x, y, z.

Proof: We shall distinguish between two cases, namely y even and y odd. We will show that in each case, the equation has no solutions.

Suppose $y = 2n$ where $n = 1, 2, \ldots$. Consider the equation

$$5^x + 103^{2n} = z^2.$$

Any power of 5 ends in the digit 5. For all values $n \geq 1$, the integer 103^{2n} ends either in the digit 9 or in the digit 1. Any power of 5 added to such an integer results in an even integer whose last digit is equal to 4 or respectively equal to 6. It is then easily seen that $5^x + 103^{2n}$ is an even integer, a multiple of 2 only. The integer z^2 is even, and therefore a multiple of at least 4. The two sides of $5^x + 103^{2n} = z^2$ then contradict each other, and hence the equation $5^x + 103^{2n} = z^2$ has no solutions.

This concludes the case $y = 2n$.

Suppose $y = 2t + 1$ where $t = 0, 1, 2, \ldots$. Consider the equation

$$5^x + 103^{2t+1} = z^2, \qquad z^2 \text{ is even.}$$

For all values $t \geq 0$, the integer 103^{2t+1} ends either in the digit 3 or in the digit 7. Any power of 5 added to such an integer ends accordingly in the integer 8 or respectively in the integer 2. Since every even square z^2 does not end in 8, nor does it end in 2, it follows that $5^x + 103^{2t+1} = z^2$ has no solutions.

The equation $5^x + 103^y = z^2$ has no solutions as asserted. □

3. The equation $5^x + 11^y = z^2$ when x, y, z are positive integers

In this section, we consider the equation $5^x + 11^y = z^2$. In Theorem 3.1, for even values y, it is established that the equation has no solutions. In Theorem 3.2, for all values $1 \leq x \leq 14$ and odd values y where $1 \leq y \leq 9$, it is determined that the equation has exactly 3 solutions.

Theorem 3.1. For all values $x \geq 1$ and $n \geq 1$, the equation $5^x + 11^{2n} = z^2$ has no solutions.

Proof: Consider the equation

$$5^x + 11^{2n} = z^2, \qquad n = 1, 2, \ldots .$$

For all values x, the integer 5^x has a last digit equal to 5, whereas for all values n, the integer 11^{2n} has a last digit equal to 1. Thus, z^2 is an integer whose last digit is equal to 6. For all values x and n, it is easily seen that $5^x + 11^{2n}$ is an even integer which is a multiple of 2 only. Since z^2 is even, it is a multiple of at least 4. The two sides of the equation $5^x + 11^{2n} = z^2$ then contradict each other implying that $5^x + 11^{2n} = z^2$ has no solutions. □

Theorem 3.2. Suppose that $1 \leq x \leq 14$ and $0 \leq t \leq 4$. Then the equation $5^x + 11^{2t+1} = z^2$ up to $5^{14} + 11^9 = 8461463316$ has 70 possibilities which yield exactly 3 solutions.

Proof: For all values $x \geq 1$, the integer 5^x has a last digit equal to 5, whereas for all values $t \geq 0$, the integer 11^{2t+1} has a last digit which equals 1. Hence, z^2 is an integer whose last digit is equal to 6. Examining each and every value $1 \leq x \leq 14$ together with each of the values $0 \leq t \leq 4$ in $5^x + 11^{2t+1}= z^2$ yields 70 possibilities, the largest of which equals $5^{14} + 11^9 = 8461463316$. Exactly 3 solutions are found which are:

Solution 1. $5^1 + 11^1 = 4^2 = z^2$,

Solution 2. $5^2 + 11^1 = 6^2 = z^2$,

Solution 3. $5^5 + 11^1 = 56^2 = z^2$.

In all other 67 possibilities of $5^x + 11^{2t+1} = z^2$, the value z is not an integer.

This completes the proof of **Theorem 3.2**. □

In view of **Theorem 3.2**, we may now raise the following conjecture.

Conjecture 1. Except for **Solutions 1 – 3**, the equation $5^x + 11^{2t+1} = z^2$ has no other solutions.

If **Conjecture 1** is indeed true, then the equation $5^x + 11^y = z^2$ has exactly 3 solutions, namely **Solutions 1- 3**.

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