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# Fixed Point Theorem using Absorbing Mappings in Fuzzy Metric Space

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***Abstract.*** In this paper, the concept of weakly semi-compatibility and sub-sequential continuity in fuzzy metric space has been applied to prove a common fixed point theorem for six self maps using implicit relation. Our result generalizes and extends the result of Ranadive and Chouhan [12].

***Keywords:*** Fuzzy metric space, common fixed point, absorbing maps, sub-compatibility, and sub-sequential continuity

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## 1. Introduction

As the theory of fuzzy sets, given by Zadeh [20] appeared in 1965, it has been used in a variety of areas of mathematics. Zadeh [21] estimated that medical diagnosis would be the most liable application domain of Fuzzy set theory. Following Zadeh's idea, Atanassov [1] introduced the concept of intuitionistic fuzzy set to permit grouping elements according to degrees of closeness and isolation. Fuzzy topology is another example of use of Zadeh's theory. George and Veeramani [4] and Kramosil and Michalek [6] have introduced the concept of fuzzy metric spaces which can be regarded as a simplification of the statistical (probabilistic) metric space. Afterwards, Grabiec [5] defined the completeness of the fuzzy metric space. Following Grabiec's work, Fang [3] further established some new fixed point theorems for contractive type mappings in G-complete fuzzy metric spaces. Soon after, Mishra et. al. [7] also obtained numerous common fixed point theorems for asymptotically commuting maps in the same space, which generalize a number of fixed point theorems in metric, Menger, $Sx_n$, $Ax_n \to p$ for some p in X as $n \to \infty$.

## 2. Definition and preliminaries

**Definition 2.6.** [15] Let A and S be mappings from fuzzy metric space (X,M,*) into itself. Then the mappings A and S are said to be semi-compatible if

$$\lim_{n\to\infty} ASx_n = Sx,$$

whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

It follows that if (A,S) is semi compatible and Ay = Sy, then ASy = SAy by taking $\{x_n\} = y$ and x = Ay = Sy.

**Definition 2.7.** [8]. A pair of maps A and B is called weakly compatible pair if they commute at their coincidence points i.e. Ax = Bx if and only if ABx = BAx.

**Definition 2.8.** [12]. Let A and B be two self maps on a fuzzy metric space (X, M, *) then A is called B-absorbing if there exists a positive integer R > 0 such that $M(Bx, BAx, t) \geq M(Bx, Ax, t/R)$ for all $x \in X$.

Similarly B is called A-absorbing if there exists a positive integer R > 0 such that $M(Ax, ABx, t) \geq M(Ax, Bx, t/R)$ for all $x \in X$.

**Preposition 2.1.** In a fuzzy metric space (X, M, *) limit of a sequence is unique.

**Preposition 2.2.** [8] If (A,S) is a semi compatible pair of self maps of a fuzzy metric space (X, M, *) and S is continuous, then (A,S) is compatible.

**Lemma 2.1.** [7] Let (X, M, *) be a fuzzy metric space. Then for all $x, y \in X$, M(x, y, .) is a non-decreasing function.

**Lemma 2.2.** [7] Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t)$ for all t > 0, then x = y.

**Lemma 2.3.** [7] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number $k \in (0, 1)$ such that $M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)$, for all t > 0 and $n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.

**Preposition 2.3.** [15] Let A and B be mappings from a fuzzy metric space (X, M, *) into itself. Assume that (A, B) is reciprocal continuous then (A, B) is semi-compatible if and only if (A, B) is compatible.

**Definition 2.9.** Self mappings A and S of a fuzzy metric space (X, M, *) are said to be weakly semi-compatible if

$$\lim_{n\to\infty} ASx_n = Sx \quad \text{or} \quad \lim_{n\to\infty} SAx_n = Ax,$$

whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Clearly, semi-compatible maps are weakly semi-compatible maps but converse is not true.

**Definition 2.10.** Self mappings A and S of a fuzzy metric space (X, M, *) are said to be sub-sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \; z \in X \text{ and satisfy}$$

$$\lim_{n\to\infty} ASx_n = Az \text{ and } \lim_{n\to\infty} SAx_n = Sz.$$

Clearly, if A and S are continuous or reciprocally continuous then they are obviously sub-sequentially continuous. However, the converse is not true in general.

**Example 2.3.** Let X = R, endowed with metric d and $M_d(x,y,t) = M(x,y,t) = \frac{t}{t+d(x,y)}$.

for all x, y∈ X , t > 0. Define the self maps A, S as

$$Ax = \begin{cases} 2, & x < 3 \\ x, & x \geq 3 \end{cases} \quad \text{and} \quad Sx = \begin{cases} 2x-4, & x \leq 3 \\ 3, & x > 3 \end{cases}.$$

Consider a sequence $\{x_n\} = 3 + \frac{1}{n}$ then

$$Ax_n = \left(3+\frac{1}{n}\right) \to 3 \text{ and } SAx_n = S\left(3+\frac{1}{n}\right) = 3 \neq S(3) = 2 \text{ as } n\to\infty.$$

Thus A and S are not reciprocally continuous but, if we consider a sequence $\{x_n\} = \left(3-\frac{1}{n}\right)$, then $Ax_n = 2$, $Sx_n = 2$, $ASx_n = 2 = A(2)$, $SAx_n = 0 = S(2)$ as $n\to\infty$.

Therefore , A and S are sub-sequentially continuous.

**Definition 2.11.** [12] A class of implicit relation

Let Φ be the set of all real continuous functions $F : (R^+)^5 \to R$ non-decreasing in first argument satisfying the following conditions :

(i) For u, v ≥ 0, F(u, v, v, u, 1) ≥ 0 implies that u ≥ v.

(ii) F(u, 1, 1, u, 1) ≥ 0 or F(u, 1, u, 1, u) ≥ 0, or F(u, u, 1, 1, u) ≥ 0 implies that u ≥ 1.

**Example 2.4.** Define $F(t_1, t_2, t_3, t_4, t_5) = 16t_1 - 12t_2 - 8t_3 + 4t_4 + t_5 - 1$. Then F ∈ Φ.

(i) $F(u, v, v, u, 1) = 20(u - v) \geq 0 \Rightarrow u \geq v$.

(ii) $F(u, 1, 1, u, 1) = 20(u - 1) \geq 0 \Rightarrow u \geq 1$ or

$F(u, 1, u, 1, u) = 9(u - 1) \geq 0 \Rightarrow u \geq 1$

or $F(u, u, 1, 1, u) = 5(u - 1) \geq 0 \Rightarrow u \geq 1$.

## 3. Main results

**Theorem 3.1.** Let A, B, S, T, P and Q be self mappings of a complete fuzzy metric space (X, M, *) with t-norm defined by a * b = min{a, b}, satisfying :

(3.1) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$;

(3.2) Q is ST-absorbing;

(3.3) for some F∈ Φ there exists q ∈ (0,1) such that for all x, y ∈ X and t > 0

$$F\{M(Px, Qy, qt), M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, qt), M(Px, STy, t)\} \geq 0.$$

(3.4) AB = BA, ST = TS, PB = BP, QT = TQ.

If the pair of maps (P, AB) is sub-sequential continuous and weakly semi-compatible then P, Q, S, T, A and B have a unique common fixed point in X.

**Proof.** Let $x_0 \in X$ be any arbitrary point. From (3.1), there exist $x_1, x_2 \in X$ such that

$$Px_0 = STx_1 \text{ and } Qx_1 = ABx_2.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Px_{2n-2} = STx_{2n-1} = y_{2n-1} \quad \text{and}$$
$$Qx_{2n-1} = ABx_{2n} = y_{2n} \quad \text{for } n = 1, 2, 3, \ldots .$$

**Step 1.** Putting $x = x_{2n}$ and $y = x_{2n+1}$ for $t > 0$ in (3.3), we get

$$F\{M(Px_{2n}, Qx_{2n+1}, qt), M(ABx_{2n}, STx_{2n+1}, t), M(Px_{2n}, ABx_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, qt), M(Px_{2n}, STx_{2n+1}, t)\} \geq 0,$$

i.e., $F\{M(y_{2n+1}, y_{2n+2}, qt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, qt), M(y_{2n+1}, y_{2n+1}, t)\} \geq 0.$

Using lemmas 2.1 and 2.2, we have

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t).$$

Again substituting $x = x_{2n+2}$ and $y = x_{2n+3}$ in (3.3), we get

$$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

Hence by lemma 2.3, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, therefore, $\{y_n\} \to z$ in X and also its subsequences converges to the same point i.e. $z \in X$,

i.e. $\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$
(1)

$$\{Px_{2n}\} \to z \qquad \{ABx_{2n}\} \to z \qquad (2)$$

**Step 2.** (P, AB) is weakly semi-compatible, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = z, \ z \in X \quad \text{and satisfy}$$
$$\lim_{n\to\infty} P(AB)x_n = ABz \quad \text{or} \quad \lim_{n\to\infty} AB(P)x_n = Pz.$$

Also, (P, AB) is sub-sequentially continuous mapping, and so

$$\lim_{n\to\infty} P(AB)x_n = Pz \text{ and } \lim_{n\to\infty} AB(P)x_n = ABz.$$

Therefore , $Pz = ABz.$ (3)

**Step 3.** Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (3.3), we have

$$F\{M(PPx_{2n}, Qx_{2n+1}, qt), M(ABPx_{2n}, STx_{2n+1}, t), M(PPx_{2n}, ABx_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, qt), M(PPx_{2n}, STx_{2n+1}, t)\} \geq 0$$

Taking $n\to\infty$ and using (1), (2), (3), we get

$$F\{M(Pz, z, qt), M(Pz, z, t), M(Pz, Pz, t), M(z, z, qt), M(Pz, z, t)\} \geq 0$$
$$F\{M(Pz, z, qt), M(Pz, z, t)\} \geq 0$$

i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$

Therefore by using lemma 2.2, we have

$$z = Pz = ABz$$

**Step 4.** Putting $x = Bz$ and $y = x_{2n+1}$ in condition (3.3), we get,

$$F\{M(PBz, Qx_{2n+1}, qt), M(ABBz, STx_{2n+1}, t), M(PBz, ABBz, t), M(Qx_{2n+1}, STx_{2n+1}, qt), M(PBz, STx_{2n+1}, t)\} \geq 0$$

As $BP = PB$, $AB = BA$, so we have

$$P(Bz) = B(Pz) = Bz \quad \text{and} \quad (AB)(Bz) = (BA)(Bz) = B(ABz) = Bz.$$

Taking $n\to\infty$ and using (1), we get

$$F\{M(Bz, z, qt), M(Bz, z, t), M(Bz, Bz, t), M(z, z, qt), M(Bz, z, t)\} \geq 0$$
$$F\{M(Bz, z, qt), M(Bz, z, t)\} \geq 0$$

i.e., $M(Bz, z, qt) \geq M(Bz, z, t).$

Therefore by using lemma 2.2, we have

$$Bz = z \text{ and also we have } ABz = Z$$

This implies $Az = z$

Therefore $Az = Bz = Pz = z$. (4)

**Step 5.** As $P(X) \subseteq ST(X)$, there exist $u \in X$ such that

$$z = Pz = STu. \tag{5}$$

Putting $x = x_{2n}$ and $y = u$ in condition (3.3), we get

$$F\{M(Px_{2n}, Qu, qt), M(ABx_{2n}, STu, t), M(Px_{2n}, ABx_{2n}, t), M(Qu, STu, qt), M(Px_{2n}, STu, t)\} \geq 0.$$

Letting $n \to \infty$ and using (2) and (5), we get

$$F\{M(z, Qu, qt), M(z, z, t), M(z, Pz, t), M(Qu, z, qt), M(z, z, t)\} \geq 0$$

As F is non-decreasing in the first argument, we have

$$F\{M(z, Qu, qt), 1, 1, M(Qu, z, qt), 1\} \geq 0$$

i.e., $M(z, Qu, qt) \geq 1$.

Therefore, $z = Qu = STu$.

Since Q is ST absorbing, we have

$$M(STu, STQu, t) \geq M(STu, Qu, t/R) \geq 1$$

i.e., $STu = STQu$ which implies $z = STz$.

Putting $x = z$ and $y = z$ in (3.3) , we get

$$F\{M(Pz, Qz, qt), M(ABz, STz, t), M(Pz, ABz, t), M(Qz, STz, qt), M(Pz, STz, t)\} \geq 0$$

or, $F\{M(z, Qz, qt), M(z, z, t), M(z, z, t), M(Qz, z, qt), M(z, z, t)\} \geq 0$.

As F is non-decreasing in the first argument, we have

$$F\{M(z, Qz, qt), 1, 1, M(Qz, z, qt), 1\} \geq 0,$$

i.e., $M(z, Qz, qt) \geq 1$.

Therefore, $z = Qz$

Hence, $z = Qz = STz$.

**Step 6.** Putting $x = x_{2n}$ and $y = Tz$ in condition (3.3), we get

$$F\{M(Px_{2n}, QTz, qt), M(ABx_{2n}, STTz, t), M(Px_{2n}, ABx_{2n}, t), M(QTz, STTz, qt), M(Px_{2n}, STTz, t)\} \geq 0$$

As $QT = TQ$ and $ST = TS$, we have

$$QTz = TQz = Tz \quad \text{and} \quad ST(Tz) = T(STz) = TQz = Tz.$$

Letting $n \to \infty$ and using (2) we get

$$F\{M(z, Tz, qt), M(z, Tz, t), M(z, z, t), M(Tz, Tz, qt), M(z, Tz, t)\} \geq 0$$

$$F\{M(z, Tz, qt), M(z, Tz, t)\} \geq 0$$

i.e., $M(z, Tz, qt) \geq M(z, Tz, t)$.

Therefore, by lemma 2.2, we get

$$Tz = z$$

Now, $STz = Tz = z$ implies $Sz = z$.

Hence, $Sz = Tz = Qz = z$.

(7)

Hence, z is the common fixed point of A, B, S, T, P and Q.

**Uniqueness:** Let w be another fixed point of A, B, P, Q, S and T. Then putting $x = z$ and $y = u$ in (3.3), we get

$$F\{M(Pz, Qu, qt), M(ABz, STu, t), M(Pz, ABz, t), M(Qu, STu, qt), M(Pz, STu, t)\} \geq 0$$

As F is non-decreasing in the first argument, we have

$$F\{M(z, u, qt), M(z, u, t), M(z, z, t), M(u, u, qt), M(z, u, t)\} \geq 0$$

or, $F\{M(z, u, qt), M(z, u, t), 1, 1, M(z, u, t)\} \geq 0$
i.e. $z = u$.
Hence z is unique fixed point in X.

**Remark 3.1.** If we take B = T = I (the identity map) in theorem 3.1, we get the following corollary.

**Corollary 3.1.** Let A, B, S, T, P and Q be self mappings of a complete fuzzy metric space (X, M, *) with t-norm defined by $a * b = \min\{a, b\}$, satisfying :

(3.1) $P(X) \subseteq S(X)$, $Q(X) \subseteq A(X)$;

(3.2) Q is S-absorbing;

(3.3) for some $F \in \Phi$ there exists $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$

$$F\{M(Px, Qy, kt), M(Ax, Sy, t), M(Px, Ax, t), M(Qy, Sy, kt), M(Px, Sy, t)\} \geq 0.$$

If the pair of maps (P, A) is sub-sequential continuous and weakly semi-compatible then P, Q, S and A have a unique common fixed point in X.

**Remark 3.2.** In view of Remark 3.1, Corollary 3.1 is a generalization of the result of Ranadive and Chouhan [12] in the sense that condition of reciprocal continuous and semi-compatible maps has been replaced by sub-sequential continuous and weakly semi-compatible maps.

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