Annals of Pure and Applied Mathematics
Vol.19, No.2, 2019, 193-196
ISSN: 2279-087X (P), 2279-0888(online)
Published on 20 May 2019
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.617v19n2a8

Annals of
Pure and Applied Mathematics

On the Diophantine Equation $53^x + 143^y = z^2$

Shivangi Asthana[1] **and** ***Madan Mohan Singh***[2]

[1]Department of Mathematics, North- Eastern Hill University, Shillong- 793022
India. E-mail: shivangiasthana.1@gmail.com
[2]Department of Basic Sciences and Social Sciences, North- Eastern Hill University
Shillong- 793022, India. E-mail: mmsingh2004@gmail.com

Received 1 April 2019; accepted 18 May 2019

Abstract. In this paper, we have shown that the Diophantine equation $53^x + 143^y = z^2$ has only two non-negative integer solutions for x, y and z. The solutions are $(0, 1, 12)$ and $(1, 1, 14)$. This equation has been solved by applying Catalan's conjecture. As a consequence of main theorem we showed that the equation $53^x + 143^y = w^4$ has no solution in non-negative integers (x, y, w).

Keywords: Exponential Diophantine equation, integer solutions

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In 2004, Mihailescu [1] proved the Catalan's conjecture: $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are non-negative integers with $\min\{a, b, x, y\} > 1$. This result plays an important role in the study of exponential Diophantine equations. In 2005, Acu [2] studied Diophantine equations of the type $a^x + b^y = c^z$ for primes a and b. In 2007, Acu [3] proved that the Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions (x, y, z) in non-negative integers. The solutions are $(3, 0, 3)$ and $(2, 1, 3)$. In 2011, Suvarnamani, Singhta and Chotchaisthit [4] showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solutions. Sroysang [5, 6] in 2012 showed that the two Diophantine equations $3^x + 5^y = z^2$ and $3^x + 17^y = z^2$ have the unique solutions $(1, 0, 2)$ in non-negative integers(x, y, z), respectively. Rabago [7] in 2013 solved the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ where x, y and z are non-negative integers. He found two solutions for each of the equations i.e., $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively. Again Sroysang [8] in 2014 solved the equation $3^x + 85^y = z^2$ and found that $(1, 0, 2)$ is a unique solution in non-negative integers x, y and z for this equation. The results on the related Diophantine equations have been found by several different mathematicians [9-14] employing a variety of methods.

In this paper, we solved the Diophantine equation

$$53^x + 143^y = z^2$$

and found that $(0, 1, 12)$ and $(1, 1, 14)$ are only two non-negative integer solutions for x, y and z.

2. Preliminaries:

We start this section by presenting a Proposition and two Lemmas.

Proposition 2.1. The Catalan's conjecture states that $(3, 2, 2, 3)$ is a solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.

Lemma 2.1. The Diophantine equation $53^x + 1 = z^2$ has no non-negative integer solution where x and z are non-negative integers.

Proof: Suppose that there are non-negative integers x and z such that $53^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is not possible. Then $x \geq 1$. Thus, $z^2 = 53^x + 1 \geq 53^1 + 1 = 54$. Then $z \geq 8$. Now we consider on the equation $z^2 - 53^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z^2 = 54$. This is a contradiction. Hence the equation $53^x + 1 = z^2$ has no non-negative integer solution.

Lemma 2.2. $(1, 12)$ is a unique solution for the (y, z) Diophantine equation $1 + 143^y = z^2$ where y and z are positive integers**.**

Proof: Let y and z be positive integers such that $1 + 143^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus $z^2 = 1 + 143^y \geq 1 + 143 = 144$. Then $z \geq 12$. Now we consider on the equation $z^2 - 143^y = 1$. By Proposition 2.1, we have $y = 1$. It follows that $z^2 = 144$. Hence, $z = 12$. Therefore, $(1, 12)$ is a unique solution (y, z) for the equation $1 + 143^y = z^2$ where y and z are positive integers.

3. Main result

Theorem 3.1. The only solutions to the Diophantine equation $53^x + 143^y = z^2$ in non-negative integers are $(0, 1, 12)$ and $(1, 1, 14)$.

Proof: The case $z = 0$ is obviously impossible. Likewise, $y = 0$ has no solution by Lemma 2.1. When $x = 0$, we have $(x, y, z) = (0, 1, 12)$ by Lemma 2.2. We consider the following remaining cases.

Case (I) $x = 1$. If $x = 1$, then we have $53^x + 143^y = z^2$. Taking $modulo\ 4$ both sides, we have $53 + 143^y = z^2 \equiv 0 \ (mod\ 4)$ that is $z = 2m$ for some natural number m. Then $53 + 143^y = 4m^2$. It follows that

$$53 + 90 + 143^y = 4m^2 + 90$$

$$\text{Or, } 143 + 143^y = 4m^2 + 90$$

$$\text{Or, } 143\left(1 + 143^{\{y-1\}}\right) = 2(2m^2 + 45)$$

So $143^{\{y-1\}} + 1 = 2$ and $2m^2 + 45 = 143$. Thus, $y = 1$ and $m^2 = 49$ which implies $m = 7$. Hence, for $m = 7$, we have $z = 14$. Therefore, $(x, y, z) = (1, 1, 14)$ is a solution to $53^x + 143^y = z^2$.

For $y = 1$ and $x = 1$, we have $z = 14$.

Case (II) $x, y, z > 1$. Suppose $53^x + 143^y = z^2$ is true for positive integers x, y and z. Here $53^x \equiv 1\ (mod\ 4)$ and $143^y \equiv 1\ (mod\ 4)$ for even integer y and $143^y \equiv 3\ (mod\ 4)$ for odd integer y. Since $z^2 \equiv 0, 1\ (mod\ 4)$ then y must be odd and z is even. We have two possibilities for x.

If x is even i.e. $x = 2l$ for some natural number$l$. Then $53^x + 143^y = z^2$ becomes $53^{2l} + 143^y = z^2$. So $(z - 53^l)(z + 53^l) = z^2 - 53^{2l} = 143^y$. So $143^{y-u} - 143^u = 2 \cdot 53^l$ where $z - 53^l = 143^u$ and $z + 53^l = 143^{\{y-u\}}, y > 2u, u$ is a non-negative integer. It follows that $143^u(143^{y-2u} - 1) = 2 \cdot 53^l$ which implies that $143^u = 1$ and $143^{y-2u} - 1 = 2 \cdot 53^l$. Thus $u = 0$ and $143^y - 1 = 2 \cdot 53^l$. Since $l \geq 1$, then $143^y = 2 \cdot 53^l + 1 \geq 2 \cdot 53^l + 1 = 107$, not possible to get a solution.

If x is odd, that is, $x = 2l + 1$ where l is a natural number. Then

$$53^x + 143^y = z^2$$

becomes $53^{2l+1} + 143^y = z^2$. So $143^y - 11 \cdot 53^{2l} = z^2 - 64 \cdot 53^{2l}$ or equivalently $143^y - 11 \cdot 53^{2l} = (z - 8 \cdot 53^l)(z + 8 \cdot 53^l)$. If $z - 8 \cdot 53^l = 1$ and $z + 8 \cdot 53^l = 143^y - 11 \cdot 53^{2l}$, then we have $16 \cdot 53^l + 11 \cdot 53^l = 143^y - 1$. So $53^l(16 + 11 \cdot 53^l) = 143^y - 1$. Clearly we see that there is no possible values for l and y to hold the equality. On the other hand, if $z - 8 \cdot 53^l = 143^y - 11 \cdot 53^{2l}$ and $z + 8 \cdot 53^l = 1$, then we have $53^l(16 + 11 \cdot 53^l) = 1 - 143^y$ not possible to get a solution. This completes the proof.

Corollary 3.1. $(0, 1, 6)$ and $(1, 1, 7)$ are exactly two non-negative integer solutions (x, y, u) for the Diophantine equation $53^x + 143^y = 4u^2$ where x, y and u are non-negative integers.

Proof: Let x, y and u be non-negative integers such that $53^x + 143^y = 4u^2$. Let $z = 2u$. Then $53^x + 143^y = z^2$. By Theorem 3.1, it follows that $(x, y, z) \in \{(0, 1, 12), (1, 1, 14)\}$. Thus, $2u = z \in \{12, 14\}$. So $u \in \{6, 7\}$. Hence, $(0, 1, 6)$ and $(1, 1, 7)$ are exactly two non-negative integer solutions (x, y, u) for the Diophantine equation $53^x + 143^y = 4u^2$.

Corollary 3.2. The Diophantine equation $53^x + 143^y = v^4$ has no non-negative integer solution where x, y and w are non-negative integers.

Proof: Suppose that there are non-negative integers x, y and w such that $53^x + 143^y = w^4$. Let $z = w^2$. Then $53^x + 143^y = z^2$. By Theorem 3.1 we have,$(x, y, z) \in \{(1, 0, 12), (1, 1, 14)\}$. Then $w^2 = z \in \{12, 14\}$. This is a contradiction.

4. Conclusion

In this paper, we have shown that the Diophantine equation $53^x + 143^y = z^2$ has only two non-negative integer solutions where x, y and z are non-negative integers. The solutions are $(0, 1, 12)$ and $(1, 1, 14)$ respectively.

Acknowledgements. The authors are highly grateful to the learned referees for their valuable suggestions for improvement of the paper.

REFERENCES

1. P.Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew.Math*., 572 (2004) 167-195.
2. D.Acu, On the Diophantine equations of type $a^x + b^y = c^z$, *Gen. Math.*, 13 (2005) 67-72.
3. D.Acu, On a Diophantine equation $2^x + 5^y = z^2$, *Gen. Math*., 15 (2007) 145-148.
4. A.Singta, S.Chotchaisthit and S.Suvarnamani, On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, Sci. *Tech. RMUTT. J.*, 1 (2011) 25-28.
5. B.Sroysang, On the Diophantine equation $3^x + 5^y = z^2$, Int. *J. Pure and Appl. Math*., 81 (2011) 605-608.
6. B.Sroysang, On the Diophantine equation $3^x + 17^y = z^2$, *Int. J. Pure and Appl. Math.*, 89 (2013) 111- 114.
7. J.F.T.Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, *Int. J. of Math. and Scientific Computing*, 3 (2013) 28-29.
8. B.Sroysang, More on the Diophantine equation $3^x + 85^y = z^2$, *Int. J. Pure and Appl. Math.*, 91 (2014) 131-134.
9. B.Sroysang, More on the Diophantine equation $2^x + 3^y = z^2$, *Int. J. Pure and Appl. Math.*, 84 (2013) 133.
10. B.Sroysang, More on the Diophantine equation $2^x + 19^y = z^2$, *Int. J. Pure Appl. Math*, 88 (2013) 157-160.
11. B.Sroysang, On the Diophantine equation $32^x + 49^y = z^2$, *J. of Mathematical Sciences: Advances and Applications*, 16 (2012) 9-12.
12. N.Burshtein, Solutions of the Diophantine Equation $p^x + (p+6)^y = z^2$ when $p,\ (p+6)$ are primes and $x + y = 2, 3, 4,$ *Annals of Pure and Applied Mathematics*, 17(1) (2018) 101- 106.
13. N.Burshtein, On solutions of the Diophantine equation $p^x + q^y = z^2$, *Annals of Pure and Applied Mathematics*, 13(1) (2017) 143- 149.
14. N.Burshtein, All the solutions of the Diophantine equation $p^3 + q^2 = z^3$, *Annals of Pure and Applied Mathematics*, 14(2) (2017) 207- 211.