Annals of Pure and Applied Mathematics
Vol. 1, No. 1, 2012, 1-12
ISSN: 2279-087X (P),2279-0888(online)
Published on 5 September 2012
www.researchmathsci.org

Annals of
Pure and Applied
Mathematics

On α-*N*-Topological Spaces Associated with Fuzzy Topological Spaces

Pradip Kumar Gain

Department of Mathematics, Kharagpur College, Inda, Kharagpur,
Paschim Medinipur-721305, West Bengal, India,
E-mail: pradip_gain@rediffmail.com

Received 8 August 2012; accepted 23 August 2012

Abstract. This paper deals with the concept of α–*limit* points of crisp subsets of a fuzzy topological space X and the concept of α–*closed* sets in X. A special type of crisp subsets (termed as α–*N-open* sets) of a fuzzy topological space is defined as the complement of α–*closed* sets in X and a special class of ordinary topological spaces is introduced. This newly defined graded topological spaces (denoted by $\tau_\alpha(N)$) are used to study and characterize some fuzzy topological properties such as compactness and other compact-like covering properties and Hausdorffness, connectedness etc.

AMS Mathematics Subject Classification (2010): **54A40; 54A05, 54D05, 54D20, 54D30**

Keywords: α–limit points, α–closed sets, α–N–open sets, α–N–topological space , α–compact, α–Hausdorff, α–Lindelöf, α–connected fuzzy topological space.

1. Introduction and Preliminaries

The concept of fuzzy sets and fuzzy set operations was introduced by L.A. Zadeh in his classical paper [15] in the year 1965, describing fuzziness mathematically for the first time. This inspired mathematicians to fuzzify mathematical structures. Subsequently several authors have applied various basic concept from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. The study of fuzzy topological spaces by considering the properties by which a space may achieve a certain degree or level was first introduced by Gantner, Steinlage and Warren [4] in 1978. They initiated a new definition of fuzzy compactness, termed as

α–compactness, by which they could assign degrees to compactness. Thereafter, several Mathematicians like Malghan and Benchalli [7], Georgion and Papadopoulos [5] have dealt with α–compactness in different ways. The approach of Gantner et al. [4] resulted into the investigation of countable α–compactness, α–Lindelöf property [8], α–Hausdorffness axiom [10], local α–compactness [7], α–closure [6] etc. in fuzzy topological spaces. In **Section 2**, we described and characterized α–compactness of Gantner et al. [4] and other compact–like covering properties such as α–almost compactness, α–*s*-closedness, α– θ*rigidityness etc. of a fuzzy topological space by means of corresponding properties of α–*N*–topological space associated with the fuzzy topological space. Countable α–compactness, α–Lindelöf property are also studied. In **Section 3**, we characterized Rodabaugh`s [10] $\alpha$–Hausdorffness of a fuzzy topological space with the help of Hausdorffness of $\alpha$–*N*–topological space. Moreover $\alpha$–Hausdorffness of a fuzzy topological space is studied using Rodabaugh`s modified definition [1]. In **Section 4**, α–connectedness in a fuzzy topological space is studied.

Throughout the present paper, by (X,τ) we mean a fuzzy topological space (fts, shortly) in Chang`s [2] sense. The class of all fuzzy sets in X is denoted by I^X. The notations ClA, $IntA$ and $1 - A$ stand respectively for the fuzzy closure [2], fuzzy interior [2] and fuzzy complement [12] of a fuzzy set A in a fts. X.

2. α-limit points, α-closed sets, α-*N*-open sets, α-*N*-topological space

In this section we propose to define a new class of crisp subsets of a fts. (X,τ) termed as α–*N*–open sets and we observed that the collection of all α–*N*–open sets in X produces a special class of ordinary topological space which is denoted by $\tau_\alpha(N)$ where $0 \leq \alpha < 1$ and termed as α–*N*–topological space. This provides gradation (fuzziness) in ordinary topological spaces. Our aspiration is to study and characterize α–compactness of Gantner et al. [4] and other compact–like covering properties by means of corresponding properties of α–*N*–topological space and vice-versa. The concept of α–limit points of an ordinary subset of a fts. and the concept of α–closed sets were introduced and studied in [3]. Changing slightly the range of α we introduced the following definitions.

***Definition* 2.1.** Let $0 \leq \alpha < 1$. Let (X, τ) be a fts and $A \subseteq X$. An element $x \in X$ is said to be an α–adherent (α–limit point) of A if for each fuzzy open set U in X with $U(x) > \alpha$, there exists $y \in A$ ($y \in A\setminus\{x\}$) such that $U(y) > \alpha$.

The set of all α–limit points of A will be denoted by A^α.

***Definition* 2.2.** A subset C of X of is said to be α–closed if it contains all its α–limit points.

***Definition* 2.3.** α–closure of a set $A \subseteq X$, denoted by $\alpha\text{–}ClA$, is defined as $\alpha\text{–}Cl\ A = A \cup A^{\alpha}$.

***Remark* 2.1.** Obviously $A \subseteq \alpha\text{–}Cl\ A$ and if $A^{\alpha} \subseteq A$ then $\alpha\text{–}ClA \subseteq A$.

A set $A \subseteq X$ is said to be α–closed iff $\alpha\text{–}Cl\ A = A$.

***Definition* 2.4.** Let $0 \leq \alpha < 1$. Let (X, τ) be a fts and $A \subseteq X$. An element $x \in X$ is said to be an α–semiadherent (α–semilimit) point of A if for each fuzzy semiopen set U in X with $\mathrm{U}(x) > \alpha$, there exists $y \in A$ ($y \in A \backslash \{x\}$) such that $\mathrm{U}(y) > \alpha$.

The set of all α–semilimit points of A will be denoted by $A^{\alpha-\mathrm{slp}}$.

***Definition* 2.5.** A subset C of X of is said to be α–semiclosed if it contains all its α–semilimit points.

***Definition* 2.6.** α–semiclosure of a set $A \subseteq X$, denoted by $\alpha\text{–}SclA$, is defined as $\alpha\text{–}Scl\ A = A \cup A^{\alpha-\mathrm{slp}}$.

A few but basic properties of α–closed sets are given by the following theorem.

***Theorem* 2.1.** Let (X, τ) be a fts and A and B be two ordinary subsets of X. Then

(a) $A \subseteq B \Longrightarrow A^{\alpha} \subseteq B^{\alpha}$ and hence $\alpha\text{–}ClA \subseteq \alpha\text{–}ClB$.
(b) $\alpha\text{–}Cl(A \cup B) = \alpha\text{–}ClA \cup \alpha\text{–}ClB$.
(c) $\alpha\text{–}Cl(A \cap B) \subseteq \alpha\text{–}ClA \cap \alpha\text{–}ClB$.
(d) the intersection of an arbitrary family of α–closed sets is a α–closed set.
(e) the union of an finite collection of α–closed sets is a α–closed set.

Proof. **(a)** Straightforward.

(b) In view of (a) $\alpha\text{–}ClA \cup \alpha\text{–}ClB \subseteq \alpha\text{–}Cl(A \cup B)$…(1). Let $x \in \alpha\text{–}Cl(A \cup B)$. Then $x \in (A \cup B)$ and $x \in (A \cup B)^{\alpha}$. If $x \in (A \cup B)$ then clearly $x \in \alpha\text{–}ClA \cup \alpha\text{–}ClB$. If $x \in (A \cup B)^{\alpha}$ then if possible let $x \notin A^{\alpha} \cup B^{\alpha}$. Then for some $V \in \tau$ with $V(x) > \alpha$, $V(y) \leq \alpha$ for all $y \in A \backslash \{x\}$ and for some $W \in \tau$ with $W(x) > \alpha$, $W(z) \leq \alpha$ for all $z \in B \backslash \{x\}$. Then $S = V \wedge W \in \tau$ with $S(x) > \alpha$ such that $S(t) \leq \alpha$ for all $t \in (A \cup B) \backslash \{x\}$ and hence $x \notin (A \cup B)^{\alpha}$ -----a contradiction.

Thus α–$Cl(A \cup B) \subseteq \alpha$–$ClA \cup \alpha$– ClB…(2). From (1) and (2) we have α–$Cl(A \cup B) = \alpha$–$ClA \cup \alpha$– ClB.

(c) Follows from (a).

(d) It is an immediate consequence of (a).

(e) Let A_1 and A_2 are two α–closed sets in X. Let $x \notin A_1 \cup A_2$. Then $x \notin A_1$ and $x \notin A_2$. As A_1 and A_2 are α–closed , it follows that x is not a α–limit point of A_1 as well as of A_2. Then for some $U \in \tau$ with $U(x) > \alpha$, $U(y) \leq \alpha$ for all $y \in A_1 \backslash \{x\}$ and for some $V \in \tau$ with $V(x) > \alpha$, $V(z) \leq \alpha$ for all $z \in A_2 \backslash \{x\}$. Then $W = U \wedge V \in \tau$ with $W(x) > \alpha$ such that $W(t) \leq \alpha$ for all $t \in (A_1 \cup A_2) \backslash \{x\}$. So x is not a α–limit point of $A_1 \cup A_2$. Hence $A_1 \cup A_2$ is α–closed.

***Definition* 2.7.** Let (X, τ) be a fts. A subset A of X is said to be α–*N*–open set in X if its complement A' is α–closed in X.

***Definition* 2.8.** Let (X, τ) be a fts. A subset B of X is said to be α–*N*–semiopen set in X if its complement B' is α–semiclosed in X.

***Theorem* 2.2.** The union of an arbitrary family of α–*N*–open sets in X is α–*N*–open set.

Proof. Let $A_1, A_1, A_1, \ldots\ldots$ be an arbitrary family of α–closed sets in a fts. X. Then $A_1', A_2', A_3', \ldots\ldots$ are all α–*N*–open sets in X. Let $x \notin (A_1' \cup A_2' \cup \ldots\ldots)'$. Then $x \notin (A_1 \cap A_2 \cap \ldots\ldots)$. As $(A_1 \cap A_2 \cap \ldots\ldots)$ is the intersection of an arbitrary collection of α–closed sets in X, $(A_1 \cap A_2 \cap \ldots\ldots\ldots)$ is α–closed. So x is not a α–limit point of $(A_1 \cap A_2 \cap \ldots\ldots)$. That is , x is not a α–limit point of $(A_1' \cup A_2' \cup \ldots\ldots\ldots)'$. So $(A_1' \cup A_2' \cup \ldots\ldots)'$ is α–closed. Hence $(A_1' \cup A_2' \cup \ldots\ldots)$ is α–*N*–open.

***Theorem* 2.3.** The intersection of a finite collection of α–*N*–open sets in X is α–*N*–open set.

Proof. Obvious.

***Remark* 2.1.** From the theorems 2.2 and 2.3 it is clear that collection of α–*N*–open sets in X in a fts. (X, τ) produces a base for an ordinary topology. We denote these topologies by $\tau_\alpha(N)$ where $(0 \leq \alpha < 1)$**.**

***Definition* 2.9.** An ordinary set X with topology $\tau_\alpha(N)$, denoted by $(X, \tau_\alpha(N))$, is called an α–*N*–topological space.

We now recall from [4] the following definitions.

***Definition* 2.10.** Let (X, τ) be a fuzzy topological space. A collection $S \subset I^X$ is called an α–shading of X, where $0 \leq \alpha < 1$ if for each $x \in X$, there exists some $U_x \in S$ such that $U_x(x) > \alpha$.

***Definition* 2.11.** A subcollection S_0 of an α–shading S of a fts (X, τ), that is also an α–shading of X is called α–subshading of S.

***Definition* 2.12.** A fts. (X, τ) is called α–compact if every α–shading of X by fuzzy open sets of X has a finite α–subshading.

***Theorem* 2.4.** Let $0 \leq \alpha < 1$. Let (X, τ) be a fts. (X, τ) is α–compact *if and only if* α–*N*–topological space $(X, \tau_\alpha(N))$ is compact.

Proof. Let (X, τ) be a fts. Let (X, τ) is α–compact. Let $\{ A_i : i \in \Lambda \}$ be a cover of X by α–N–open subsets A_i of X. That is, $X \subseteq \bigcup_{i\in\Lambda} A_i$, $A_i \subseteq X$, $i \in \Lambda$. Let $x_1 \in X \Longrightarrow x_1 \in A_i$ for some $i \in \Lambda$. Let $x_1 \in A_1$. Then $x_1 \notin A_1{}'$. As A_1 is α–N–open, $A_1{}'$ is α–closed. So x_1 is not a α–limit point of $A_1{}'$. Then there exists a fuzzy open set U_{x_1} in X with $U_{x_1}(x_1) > \alpha$ and $U_{x_1}(y) \leq \alpha$ for all $y \in A_1{}'$. Similarly, for $x_2 \in X$, we must have a fuzzy open set U_{x_2} in X with $U_{x_2}(x_2) > \alpha$ and $U_{x_2}(y) \leq \alpha$ for all $y \in A_2{}'$ (say). Proceeding in this way, we must have a collection $F = \{ U_{x_i} : x_i \in X; \ i \in \Lambda \}$ of fuzzy open sets in X such that $U_{x_i}(x_i) > \alpha$, $i = 1,2,3,\ldots\ldots$. Obviously F is a α–shading of X by fuzzy open sets U_{x_i} in X. As X is α–compact, there must be a finite subcollection $F' = \{ U_{x_i} : x_i \in X, \ i = 1,2,3,\ldots\ldots,n\}$ of F such that $U_{x_i}(x_i) > \alpha$ for each $x_i \in X$, $i = 1,2,3,\ldots\ldots,n$. Now for each $x_i \in X$, $x_i \in A_i$ $i = 1,2,3,\ldots\ldots, n$ So $X \subseteq \bigcup_{i=1}^{n} A_i$. Therefore, $\{ A_i , i = 1,2,3,\ldots\ldots,n\}$ is a finite subcover of X by α–*N*–open subsets A_i of X. Hence $(X, \tau_\alpha(N))$ is compact. Conversely, let $(X, \tau_\alpha(N))$ is compact. Let $A = \{ A_1, A_2,\ldots\ldots\}$ be a cover of X by α–*N*–open subsets $A_1, A_2, \ldots\ldots$of X. That is, $X \subseteq \bigcup_{i\in\Lambda} A_i$. Let $x \in X$. Then $x \in A_i$ for some $i \in \Lambda$. Let $x \in A_1$. Then $x \notin A_1{}'$. As A_1 is α–*N*–open, $A_1{}'$ is α–closed. So x is not a α–limit point of $A_1{}'$. Then there must exists a fuzzy open set U_x in X with $U_x(x) > \alpha$ such that $U_x(p) \leq \alpha$ for all $p \in A_1{}'$. Similarly, for $y \in X$, we must have a fuzzy open set U_y in X with $U_y(y) > \alpha$ such that $U_y(z) \leq \alpha$ for all $z \in A_2{}'$ (say). Proceeding in this way, we must have a collection $F = \{ U_x : x \in X \}$ of fuzzy open sets in X

such that $U_x(x) > \alpha$ for all $x \in X$. Clearly F is a α–shading of X. As $(X, \tau_\alpha(\mathrm{N}))$ is compact there must be a finite subcollection B of A such that $X \subseteq \bigcup_{i=1}^{n} B_i$, $B_i \in B, i = 1,2,3,\ldots\ldots\ldots, n.$ As $B \subseteq A$, we must have a finite subcollection $F' = \{U_{x_i} : x_i \in X,\ i = 1,2,3,\ldots\ldots,n\}$ of F such that $U_{x_i}(x_i) > \alpha$ for each $x_i \in X,\ i = 1,2,3,\ldots\ldots\ldots,n.$ So F' is a finite α–subshading of X. Hence (X, τ) is α–compact.

Countably compactness of an fts. has been studied in [8],[11],[13].

***Definition* 2.13.** [1] Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be countably α–compact if every countable α–shading of X by fuzzy open sets in X has a finite α–subshading.

It is easy to verify the following.

***Theorem* 2.5.** Let $0 \leq \alpha < 1$. Let (X, τ) be a fts. (X, τ) is countable α–compact *if and only if* α–*N*–topological space $(X, \tau_\alpha(N))$ is countable compact topological space.

Lindelöf fuzzy topological spaces were studied in [8],[9],[14].

Lindelöf fuzzy topological spaces, using shading family are given below.

Definition 2.14. [1] Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be α–Lindelöf *if and only if* every α–shading of X by fuzzy open sets in X has a countable α–subshading.

It is easy to verify the following.

***Theorem* 2.6.** Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be α–Lindelöf *if and only if* $(X, \tau_\alpha(N))$ is Lindelöf topological space.

We now set the following definitions compact-like covering properties for a fts.

***Definition* 2.15.** Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be α–*AC* (resp. α–*s*-closed) space if for every α–shading S of X by fuzzy open (resp. fuzzy semiopen) sets of X, there exists a finite subset S_0 of S such that for each $x \in X$ there exists $U \in S_0$ such that $ClU(x) > \alpha$ (resp. $SclU(x) > \alpha$).

It is easy to verify the following.

***Theorem* 2.7.** Let $0 \leq \alpha < 1$. A fts. (X, τ) is α–AC space *if and only if* α–*N*–topological space $(X, \tau_\alpha(N))$ is almost compact.

***Theorem* 2.8.** Let $0 \leq \alpha < 1$. A fts. (X, τ) is α–s-closed space *if and only if* α–*N*–topological space $(X, \tau_\alpha(N))$ is s-closed.

***Definition* 2.16.** Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be α–θ*rigid space if for every α–shading S of X by fuzzy semiopen sets of X, there exists a finite subset S_0 of S such that for each $x_i \in X$ there exists $U_{x_i} \in S_0$ such that $Scl(\bigvee_{i=1}^{n} U_{x_i}(x_i)) > \alpha$, $x_i \in X$.

***Theorem* 2.9.** Let $0 \leq \alpha < 1$. A fts. (X, τ) is α–θ*rigid space *if and only if* α–*N*–topological space $(X, \tau_\alpha(N))$ is θ*rigid.

Proof. Let (X, τ) be a fts. Let (X, τ) is α–θ*rigid space. Let $\{ A_i : i \in \Lambda \}$ be a cover of X by α–*N*–semiopen subsets A_i of X. That is, $X \subseteq \bigcup_{i\in\Lambda} A_i$, $A_i \subseteq X, i \in \Lambda$. Let $x_1, x_2, x_3, \dots \dots \in X$. Then by the same argument given in the proof of the theorem 2.4, we must have a collection $F = \{ U_{x_i} : x_i \in X;\ i \in \Lambda \}$ of fuzzy semiopen sets U_{x_i} in X such that $U_{x_i}(x_i) > \alpha$ for each each $x_i \in X$, $i =$ 1,2,3,………. and F is clearly a α–shading of X by fuzzy semiopen sets in X. As (X, τ) is α–θ*rigid space, there must be finite subcollection $F' = \{ U_{x_i} : x_i \in X$, $i =$ 1,2,3,……,n $\}$ of F such that $Scl(\bigvee_{i=1}^{n} U_{x_i}(x_i)) > \alpha$ for each $x_i \in X$, $i =$ 1,2,3,……..,n. Now for each $x_i \in X$, $x_i \in A_i$, $i =$1,2,3,…..…,n. Therefore, $x_i \in \{\alpha\text{–}Scl\ A_i\}$, $i =$ 1,2,3,……,n. So $X \subseteq \bigcup_{i=1}^{n} A_i$. Therefore,,
$X \subseteq \alpha\text{–}Scl\,(\bigcup_{i=1}^{n} A_i)$. Hence $(X, \tau_\alpha(N))$ is θ*rigid.
Conversely, let $(X, \tau_\alpha(N))$ is θ*rigid. Let $A = \{ A_1, A_2, \dots\dots\}$ be a cover of X by α–*N*–semiopen subsets $A_1, A_2, \dots$of X. That is, $X \subseteq \bigcup_{i\in\Lambda} A_i$. Let $, y, z, \dots \in X$. Then by the same argument given in the proof of the theorem 2.4, we must have a collection $F = \{ U_x : x \in X\}$ of fuzzy semiopen sets in X such that $U_x(x) > \alpha$ for all $x \in X$. …….(1). Clearly F is a α–shading of X by fuzzy semiopen sets of X. As $(X, \tau_\alpha(N))$ is θ*rigid, there must be a finite subcollection B of A of α –*N*–semiopen sets in X such that $X \subseteq \alpha\text{–}Scl\,(\bigcup_{i=1}^{n} B_i)$, $B_i \in B$. As $x_i \in X$, $x_i \in$ α–$Scl\,(\bigcup_{i=1}^{n} B_i)$ for some $i =$ 1,2,3,…….,n which implies that x_i is a α–semiadherent point of $(\bigcup_{i=1}^{n} B_i)$ for some $i =$ 1,2,3,……,n. Then for $x_i \in X$ there exists fuzzy semiopen set U_{x_i} in X with $U_{x_i}(x_i) > \alpha$ such that $U_{x_i}(p) > \alpha$ for all $p \in (\bigcup_{i=1}^{n} B_i)$, $i =$1,2,3,……,n. (by(1)). Thus we get a finite subcollection $F' = \{ U_{x_i} : x_i \in X, i =$ 1,2,3,…,$n\}$ of F such that $U_{x_i}(x_i) > \alpha$, $i =$ 1,2,3,……,n. As $U_{x_i}(x_i) > \alpha$, $(\bigvee_{i=1}^{n} U_{x_i})(x_i) > \alpha$, $i =$1,2,3,………,n. That is, $Scl(\bigvee_{i=1}^{n} U_{x_i}(x_i)) > \alpha, x_i \in X, i =$ 1,2,3,……,n. Hence X is α–θ*rigid.

3. α–Hausdorffness of fuzzy topological spaces

***Definition* 3.1.** [10] Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be α–Hausdorff if for each $x, y \in X$ with $x \neq y$, there exist fuzzy open sets A and B in X, such that $A(x) > \alpha$, $B(y) > \alpha$ and $A \wedge B = 0_X$.

***Theorem* 3.1.** Let $0 \leq \alpha < 1$. If a fts. (X, τ) is α–Hausdorff then α–*N*–topological space $(X, \tau_\alpha(N))$ is Hausdorff.

Proof. Let (X, τ) is α–Hausdorff. Let $x, y \in X$ with $x \neq y$. Then there must exist $A, B \subseteq X$ such that $x \in A$, $y \in B$ and $A \cap B = \phi$. By α–Hausdorffness of X, there exist fuzzy open sets U and V in X such that $U(x) > \alpha$, $V(y) > \alpha$ and $U \wedge V = 0_X$. Therefore, $inf(U(x), V(x)) = 0$, $\forall x \in X$. By α –Hausdorffness of X, $U(x) > \alpha$. So $V(x) = 0$, for some $x \in X$. That is, $U(x) > \alpha$, $V(x) \leq \alpha$ (since, $0 \leq \alpha < 1$)……..(1). Also $inf(U(y), V(y)) = 0$, $\forall y \in X$. By α –Hausdorffness of X, $V(y) > \alpha$. So $U(y) = 0$, for some $y \in X$. That is, $V(y) > \alpha$, $U(y) \leq \alpha$ (since, $0 \leq \alpha < 1$)……..(2). By (1) and (2), we have $U(x) > \alpha$, $U(y) \leq \alpha$……(3) and $V(y) > \alpha$, $V(x) \leq \alpha$…(4). From (3) it follows that x is not a α–limit point of B. So B is α–closed and B' is α–*N*–open. As $x \in A$ and $A \cap B = \phi$, it follows that $x \in B'$ .Similarly, from (4) it follows that y is not a α–limit point of A. So A is α–closed and A' is α–*N*–open. As $y \in B$ and $A \cap B = \phi$, it follows that $y \in A'$. Thus there exist two α–*N*–open sets $B'$and A' such that for two distinct elements x and y in X, $x \in B'$, $y \in A'$ and $B' \cap A' = \phi$. Hence $(X, \tau_\alpha(N))$ is Hausdorff.

Converse of the above theorem holds only for $\alpha = 0$ which is justified by the following theorem.

***Theorem* 3.2.** Let (X, τ) be a fts. If 0–*N*–topological space $(X, \tau_0(\mathrm{N}))$ is Hausdorff then (X, τ) is 0–Hausdorff.

Proof. Let $(X, \tau_0(N))$ is Hausdorff. Then for every distinct points $x_1, x_2 \in X$, there exist 0–*N*–open sets A_1, A_2 in X such that $x_1 \in A_1$, $x_2 \in A_2$and $A_1 \cap A_2 = \phi$. As $x_1 \in A_1$, $x_1 \notin A_1{}'$. As A_1 is 0–*N*–open, $A_1{}'$ is 0–closed. So x_1 is not a 0–limit point of $A_1{}'$. Then there exists a fuzzy open set U_1 in X with $U_1(x_1) > 0$ and $U_1(y)= 0$ for all $y \in A_1{}'$. That is, $U_1(x_1) > 0$ and $U_1(x_2) = 0$ (since $x_2 \in A_1{}'$)……..(1). Similarly, as $x_2 \in A_2$, we must have a fuzzy open set U_2 in X with $U_2(x_2) > 0$ and $U_2(z) = 0$ for all $z \in A_2{}'$. That is, $U_2(x_2) > 0$ and $U_2(x_1) = 0$ (since $x_1 \in A_2{}'$). ……..(2). From (1) and (2) we have, $U_1(x_1) > 0$, $U_2(x_1)= 0$ and $U_2(x_2) > 0$, $U_1(x_2)= 0$. $x_1, x_2 \in X$. That is, $inf(U_1(x), U_2(x)) = 0$, $\forall x \in X$. This implies $U_1 \wedge U_2 = 0_X$. Hence (X, τ) is 0–Hausdorff.

***Definition* 3.2.** [1] Let X be a set and $0 \leq \alpha < 1$. A family $\{A_\alpha\}_{\alpha\in\Lambda}$ of fuzzy sets in X is said to be α–disjoint if $(\wedge_{\alpha\in\Lambda} A_\alpha)(x) \leq \alpha$.

It is evident from the above definition that two fuzzy sets A and B in X are α–disjoint if and only if for each $x \in X$ either $A(x) \leq \alpha$ or $B(x) \leq \alpha$.

Rodabaugh`s [10] definition of α–Hausdorffness can suitably modified in the following way.

***Definition* 3.3.** [1] Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be α–Hausdorff if for each $x, y \in X$ with $x \neq y$, there exist fuzzy open sets A and B in X, such that $A(x) > \alpha$, $B(y) > \alpha$ and A and B are α–disjoint.

For the modified class of α–Hausdorff fuzzy topological spaces, we have the following theorem.

***Theorem* 3.3.** Let $0 \leq \alpha < 1$. A fts. (X, τ) is α–Hausdorff *if and only if* α–*N*–topological space $(X, \tau_\alpha(N))$ is Hausdorff.

Proof. Let (X, τ) is α–Hausdorff. Let $x,\ y \in X$ with $x \neq y$. Then there must exist $A, B \subseteq X$ such that $x \in A$, $y \in B$ and $A \cap B = \phi$. By α–Hausdorffness (modified definition) of X, there exist fuzzy open sets U and V in X such that $U(x) > \alpha$, $V(y) > \alpha$ and $U \wedge V \leq \alpha$. Therefore, $inf(U(x),\ V(x)) \leq \alpha$, $\forall x \in X$. By α–Hausdorffness of X, $U(x) > \alpha$. So $V(x) \leq \alpha$, for some $x \in X$. That is, $U(x) > \alpha$, $V(x) \leq \alpha$ (since, $0 \leq \alpha < 1$)……..(1). Also $inf(U(y),\ V(y)) \leq \alpha$, $\forall y \in X$. By α –Hausdorffness of X, $V(y) > \alpha$. So $U(y) \leq \alpha$, for some $y \in X$. That is, $V(y) > \alpha$, $U(y) \leq \alpha$ (since, $0 \leq \alpha < 1$)……..(2). By (1) and (2), we have $U(x) > \alpha, U(y) \leq \alpha$……(3) and $V(y) > \alpha, V(x) \leq \alpha$…(4). From (3) it follows that x is not a α–limit point of B. So B is α–closed and B' is α–*N*–open. As $x \in A$ and $A \cap B = \phi$, it follows that $x \in B'$. Similarly, from (4) it follows that y is not a α–limit point of A. So A is α–closed and A' is α–*N*–open. As $y \in B$ and $A \cap B = \phi$, it follows that $y \in A'$. Thus there exist two α–N–open sets $B'$and A' such that for two distinct elements x and y in X, $x \in B', y \in A'$and $B' \cap A' = \phi$. Hence $(X, \tau_\alpha(N))$ is Hausdorff.

Conversely, let $(X, \tau_\alpha(N))$ is Hausdorff. Then for every distinct points $x_1, x_2 \in X$, there exist α–*N*–open sets A_1, A_2 in X such that $x_1 \in A_1, x_2 \in A_2$and $A_1 \cap A_2 = \phi$. As $x_1 \in A_1$, $x_1 \notin A_1{}'$. As A_1 is α–*N*–open, $A_1{}'$ is α–closed. So x_1 is not a α–limit point of $A_1{}'$. Then there exists a fuzzy open set U_1 in X with $U_1(x_1) > \alpha$ and $U_1(y) \leq \alpha$ for all $y \in A_1{}'$. That is, $U_1(x_1) > \alpha$ and $U_1(x_2) \leq \alpha$ (since

$x_2 \in {A_1}'$)……..(1). Similarly, as $x_2 \in A_2$, we must have a fuzzy open set U_2 in X with $U_2(x_2) > \alpha$ and $U_2(z) \leq \alpha$ for all $z \in {A_2}'$. That is, $U_2(x_2) > \alpha$ and $U_2(x_1) \leq \alpha$ (since $x_1 \in {A_2}'$)……..(2). From (1) and (2) we have, $U_1(x_1) > \alpha$, $U_2(x_1) \leq \alpha$ and $U_2(x_2) > \alpha$, $U_1(x_2) \leq \alpha$. $x_1, x_2 \in X$. That is, $inf(U_1(x), U_2(x)) \leq \alpha$, $\forall x \in X$. This implies $U_1 \wedge U_2 \leq \alpha$. Therefore U_1 and U_2 are α–disjoint. Hence (X, τ) is α–Hausdorff.

4. α–Connected fuzzy topological spaces

***Definition* 4.1.** [1] Let $0 \leq \alpha < 1$. Let X be a non-empty set. A fuzzy set A in X is said to be an empty set of order α if $A(x) \leq \alpha$ for each $x \in X$.

A fuzzy set A in X is said to be non-empty of order α if there exists $x_0 \in X$ such that $A(x_0) > \alpha$.

***Definition* 4.2.** [1] Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be α–disconnected if there exists an α–shading of two fuzzy open sets in X which are non-empty of order α and α–disjoint.

***Definition* 4.3.** [1] Let $0 \leq \alpha < 1$. A fts. (X, τ) is said to be α–connected if there does not exists an α–shading of two fuzzy open sets in X which are non-empty of order α and α–disjoint.

***Theorem* 4.1.** Let $0 \leq \alpha < 1$. A fts. (X, τ) is α–connected *if and only if* α–N–topological space $(X, \tau_\alpha(\mathrm{N}))$ is connected.

Proof. Let (X, τ) is α–connected. If possible let $(X, \tau_\alpha(N))$ is disconnected. Then there exist non-empty disjoint α–*N*–open sets A and B of X such that $A \cup B = X$. So A' and B' are α–closed, disjoint and $A' \cup B' = X$. Let $x \in A'$, $y \in B'$. Clearly, y is not a α–limit point of A'. Then there exists a fuzzy open set U_1 in X with $U_1(\mathrm{y}) > \alpha$ and $U_1(x) \leq \alpha$ for all $x \in A'$. Similarly, as x is not a α–limit point of B', there exists a fuzzy open set U_2 in X with $U_2(\mathrm{x}) > \alpha$ and $U_2(y) \leq \alpha$ for all $y \in B'$. Thus we get a family $F = \{ U_1, U_2\}$ of two fuzzy open sets such that $U_1(\mathrm{y}) > \alpha$, $U_2(\mathrm{x}) > \alpha$, $x, y \in X$. So $F = \{ U_1, U_2\}$ is an α–shading of two fuzzy open sets in X. Now $U_2(\mathrm{x}) > \alpha$, $U_1(x) \leq \alpha$ and $U_1(\mathrm{y}) > \alpha$, $U_2(y) \leq \alpha$, $x, y \in X$. That is, $Min\{U_1(x), U_2(\mathrm{x})\} \leq \alpha$, $x \in X$. That is, $(U_1 \wedge U_2)(x) \leq \alpha$. Therefore, U_1 and U_2 are α–disjoint. As there exist $x \in X$, $y \in X$ such that $U_1(\mathrm{y}) > \alpha$, $U_2(\mathrm{x}) > \alpha$ and this implies U_1 and U_2 are non-empty fuzzy sets of order α. Thus $F = \{ U_1, U_2\}$ is an α–shading of two non-empty fuzzy open sets U_1 and U_2 of order α and which are α–disjoint. So (X, τ) is α–disconnected, which contradicts the hypothesis. Hence $(X, \tau_\alpha(\mathrm{N}))$ is connected.

Conversely, let $(X, \tau_\alpha(N))$ is connected. If possible let (X, τ) is α–disconnected. There exists an α–shading $F = \{U_1, U_2\}$ of two non-empty fuzzy open sets U_1 and U_2 in X of order α and which are α–disjoint. So there exist $x_0 \in X$ and $x_0' \in X$ such that $U_1(x_0) > \alpha, U_2(x_0) > \alpha$, and for each $x \in X$, $(U_1 \wedge U_2)(x) \leq \alpha$, ie, *Min*$\{U_1(x), U_2(\mathrm{x})\} \leq \alpha$, $x \in X$………(1). As $F = \{U_1, U_2\}$ is an α–shading of X, for each $x \in X$, either $U_1(x) > \alpha$ or $U_2(\mathrm{x}) > \alpha$. Let $x = x_0$. Then either $U_1(x_0) > \alpha$ or $U_2(x_0) > \alpha$. Since $U_1(x_0) > \alpha$, $U_2(x_0) \leq \alpha$ (by (1)). Again let $x = x_0'$, then either $U_1(x_0') > \alpha$ or $U_2(x_0') > \alpha$. Since $U_2(x_0') > \alpha$, $U_1(x_0') \leq \alpha$ (by(1)). Let $x_0 \in A \subset X$ and $x_0' \in B \subset X$ with $A \cap B = \phi$ such that $A \cup B = X$. Now $U_1(x_0) > \alpha$, $U_1(x_0') \leq \alpha$ implies x_0 is not a α–limit point of B. So B is α–closed. Hence B' is α–open. Again $U_2(x_0') > \alpha$ and $U_2(x_0) \leq \alpha$ implies x_0' is not a α–limit point of A. So A is α–closed. Hence A' is α–open. Also $A' \cap B' = \phi$ and $A' \cup B' = X$ (since $A \cap B = \phi$ and $A \cup B = X$). Thus there exist two non-empty disjoint α–*N*–open sets $A'$and B' in X such that $A' \cup B' = X$. Hence $(X, \tau_\alpha(\mathrm{N}))$ is disconnected which contradicts the hypothesis. Hence (X, τ) is α–connected.

Acknowledgment

The author is thankful to the University Grants Commission, New Delhi, for sponsoring this work under grant of Minor Research Project..

REFERENCES

1. S. S. Benchalli and G. P. Siddapur; On the level spaces of fuzzy topological spaces, *Bulletin of Mathematical Analysis and Applications,* 1(2) (2009), 57-65.
2. C. L. Chang; Fuzzy topological spaces, *J. Math.Anal. Appl,* 24 (1968), 182-190.
3. R. P. Chakraborty; On some problems associated with fuzzy topology, *Ph.D Dissertation,*University of Calcutta, 2001.
4. T. E. Gantner, R. C. Steinlage and R. H. Warren; compactness in fuzzy topological spaces,*J. Math.Anal. Appl,* 62 (1978), 547-562.
5. D. N. Georgiou and B. K. Papadopoulos; On fuzzy compactness, *J. Math.Anal. Appl,* 233 (1999), 86-101.
6. A. J. Klein, α–closure in fuzzy topology, Rocky Mount. *J. Math,* 11 (1981), 553-560.
7. S. R. Malghan and S. S. Benchalli; Open maps, closed maps and local compactness in fuzzy topological spaces, *J. Math.Anal. Appl,* 99(2) (1984), 338-349.

8. S. R. Malghan and S. S. Benchalli; On fuzzy topological spaces, *Glasnik Mat,* 16(36) (1981), 313-325.
9. P. M. Pu and Y. M. Liu; Fuzzy topology,I.neighbourhood structure of a fuzzy point and Moor-Smith convergence, *J.Math. Anal. Appl.*76(2) (1980), 571-599.
10. S. E. Rodabaugh; The Hausdorff separation axiom for fuzzy topological spaces, *Topology and its appl.* 11 (1980), 319-334.
11. M. Sarkar; On fuzzy topological spaces, *J. Math.Anal. Appl,* 79 (1981), 384-394.
12. S. P. Sinha and S. Malakar; On S-closed fuzzy topological spaces,*J.Fuzzy Math,* 2(1) (1994), 95-103.
13. C. K. Wong; Fuzzy topology; product and quotient theorems, *J. Math.Anal. Appl,* 45 (1974), 512-521.
14. C. K. Wong; Covering properties of fuzzy topological spaces, *J. Math.Anal. Appl,* 43 (1973), 697-704.
15. L. A. Zadeh; Fuzzy sets, *Information and Control,* 8 (1965), 338-353.