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An Equation Related to Centralizers in Semiprime Gamma Rings

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Abstract. Let M be a 2-torsion free semiprime Γ-ring satisfying a certain assumption and let $T: M \rightarrow M$ be an additive mapping such that

$$2T(a\alpha b\beta a) = T(a)\alpha b\beta a + a\alpha b\beta T(a)$$

holds for all pairs $a, b \in M$, and $\alpha, \beta \in \Gamma$. Then we prove that T is a centralizer.

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1. Introduction

Let M and Γ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \rightarrow x\alpha y$ of $M \times \Gamma \times M \rightarrow M$, which satisfies the conditions

(i) $x\alpha y \in M$

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

then M is called a Γ-ring.

Every ring M is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa[13].

Bernes[1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa.

Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U$ ($U\Gamma M \subset U$). If U is both a left and a right ideal, then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2x=0$ implies $x=0$ for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b=(0)$ with $a,b \in M$, implies $a=0$ or $b=0$ and semiprime if $a\Gamma M\Gamma a=(0)$ with $a \in M$ implies $a=0$. Furthermore, M is said to be commutative Γ-ring if $x\alpha y = y\alpha x$ for all $x,y \in M$ and $\alpha \in \Gamma$. Moreover,the set $Z(M)$ $=\{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma, y \in M\}$ is called the centre of the Γ-ring M.

If M is a Γ-ring, then $[x,y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x,y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$$[x\alpha y, z]_\beta = [x,z]_\beta \alpha y + x[\alpha,\beta]_z y + x\alpha[y,z]_\beta$$

and $[x, y\alpha z]_\beta = [x,y]_\beta \alpha z + y[\alpha,\beta]_x z + y\alpha[x,z]_\beta$,

for all $x, y.z \in M$ and $\alpha,\beta \in \Gamma$. We consider the following assumption:

$$x\alpha y\beta z = x\beta y\alpha z \text{, for all } x,y,z \in M \text{, and } \alpha,\beta \in \Gamma. \qquad (A)$$

According to the assumption (A), the above two identites reduce to

$$[x\alpha y, z]_\beta = [x,z]_\beta \alpha y + x\alpha[y,z]_\beta$$

and $[x, y\alpha z]_\beta = [x,y]_\beta \alpha z + y\alpha[x,z]_\beta$, which we extensively used.

An additive mapping $T: M \to M$ is a left(right) centralizer if $T(x\alpha y) = T(x)\alpha y$ $(T(x\alpha y) = x\alpha T(y))$ holds for all $x,y \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T(x) = a\alpha x$ is a left centralizer and $T(x) = x\alpha a$ is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers. An additive mapping $D: M \to M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x,y \in M$, and $\alpha \in \Gamma$ and is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

An additive mapping $T: M \to M$ is Jordan left(right) centralizer if $T(x\alpha x) = T(x)\alpha x (T(x\alpha x) = x\alpha T(x))$ for all $x \in M$, and $\alpha \in \Gamma$.

Every left centralizer is a Jordan left centralizer but the converse is not ingeneral true.

An additive mappings $T: M \to M$ is called a Jordan centralizer if $T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Bernes[1], Luh [6] and Kyuno[5] studied the structure of Γ-rings and obtained various generalizations of corresponding parts in ring theory.

Borut Zalar [12] worked on centralizers of semiprime rings and proved that Jordan centralizers and centralizers of this rings coincide. Joso Vukman[9,10,11] developed some remarkable results using centralizers on prime and semiprime rings.

Vukman and Irena [8] proved that if R is a 2-tortion free semiprime ring and $T: R \to R$ is an additive mapping such that $2T(xyx) = T(x)yx + xyx$ holds for all $x, y \in R$, then T is a centralizer.

Y.Ceven [2] worked on Jordan left derivations on completely prime Γ-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ-ring that makes the Γ-ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime Γ-ring is a left derivation on it.

In [3], M. F. Hoque and A.C Paul have proved that every Jordan centralizer of a 2-torsion free semiprime Γ-ring is a centralizer. There they also gave an example of a Jordan centralizer which is not a centralizer.

In [4], M. F. Hoque and A.C Paul have proved that if M is a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and if $T: M \to M$ is an additive mapping such that $T(x\alpha y\beta x) = x\alpha T(y)\beta x$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then T is a centralizer. Also, they have proved that T is a centralizer if M contains a multiplicative identity 1.

In this paper, we devolep some results of [8] in Γ-rings by assuming an assumption (A). Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and let $T: M \to M$ be an additive mapping such that

$$2T(a\alpha b\beta a) = T(a)\alpha b\beta a + a\alpha b\beta T(a) \qquad (1)$$

holds for all pairs $a, b \in M$, and $\alpha, \beta \in \Gamma$. Then T is a centralizer.

2. The Centralizers of Semiprime Gamma Rings

For proving our main results, we need the following Lemmas:

Lemma 2.1. *Suppose M is a semiprime Γ-ring satisfying the assumption (A). Suppose that the relation $x\alpha a\beta y + y\alpha a\beta z = 0$ holds for all $a \in M$, some $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then $(x+z)\alpha a\beta y = 0$ is satisfied for all $a \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. The proof of this lemma can be founded in ([4],Lemma 2.1).

Lemma 2.2. *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and let $T: M \to M$ be an additive mapping. Suppose that*

$$2T(a\alpha b\beta a) = T(a)\alpha b\beta a + a\alpha b\beta T(a)$$

holds for all pairs $a, b \in M$ and $\alpha, \beta \in \Gamma$. Then $2T(a\gamma a) = T(a)\gamma a + a\gamma T(a)$.

Proof. Putting $a+c$ for a in (1)(linearization), we have

$$2T(a\alpha b\beta c + c\alpha b\beta a) = T(a)\alpha b\beta c + T(c)\alpha b\beta a + c\alpha b\beta T(a) + a\alpha b\beta T(c) \quad (2)$$

Putting $c = a\gamma a$ in (2), we have

$$2T(a\alpha b\beta a\gamma a + a\gamma a\alpha b\beta a)$$
$$= T(a)\alpha b\beta a\gamma a + T(a\gamma a)\alpha b\beta a + a\gamma a\alpha b\beta T(a) + a\alpha b\beta T(a\gamma a) \quad (3)$$

Replacing b by $a\gamma b + b\gamma a$ in (1), we have

$$2T(a\alpha a\gamma b\beta a + a\alpha b\gamma a\beta a)$$
$$= T(a)\alpha a\gamma b\beta a + T(a)\alpha b\gamma a\beta a + a\alpha a\gamma b\beta T(a) + a\alpha b\gamma a\beta T(a) \quad (4)$$

Subtracting (4) from (3), using assumption (A), gives

$$(T(a\gamma a) - T(a)\gamma a)\alpha b\beta a + a\alpha b\beta(T(a\gamma a) - a\gamma T(a)) = 0$$

Taking $x = T(a\gamma a) - T(a)\gamma a$, $y = a$, $c = b$ and $z = T(a\gamma a) - a\gamma T(a)$. Then the above relation becomes $x\alpha c\beta y + y\alpha c\beta z = 0$. Thus using Lemma 2.1, we get $(x+z)\alpha c\beta y = 0$. Hence $(2T(a\gamma a) - T(a)\gamma a - a\gamma T(a))\alpha b\beta a = 0$.

If we take $A(a) = 2T(a\gamma a) - T(a)\gamma a - a\gamma T(a)$, then the above relation becomes

$$A(a)\alpha b\beta a = 0$$

Using the assumption (A), We obtain

$$A(a)\beta b\alpha a = 0 \quad (5)$$

Replacing b by $a\alpha b\gamma A(a)$ in (5), we have $A(a)\beta a\alpha b\gamma A(a)\alpha a = 0$

Again using the assumption (A), we have $A(a)\alpha a\beta b\gamma A(a)\alpha a = 0$

By the semiprimeness of M, we have

$$A(a)\alpha a = 0 \quad (6)$$

Similarly, if we multiplying (5) from the left by $a\alpha$ and from the right side by $\gamma A(a)$, we obtain $a\alpha A(a)\beta b\alpha a\gamma A(a) = 0$

Using the assumption (A), $a\alpha A(a)\beta b\gamma a\alpha A(a) = 0$ and by the semiprimeness, we obtain

$$a\alpha A(a) = 0 \quad (7)$$

Replacing a by $a+b$ in (6)(linearization), we have

$$A(a)\alpha b + A(b)\alpha a + B(a,b)\alpha a + B(a,b)\alpha b = 0,$$

where $B(a,b) = 2T(a\gamma b + b\gamma a) - T(a)\gamma b - T(b)\gamma a - a\gamma T(b) - b\gamma T(a)$

Replacing a by $-a$ in the above relation and comparing these relation, and by using the 2-tortion freeness of M, we arrive at

$$A(a)\alpha b + B(a,b)\alpha a = 0 \quad (8)$$

Right multiplication of the above relation by $\beta A(a)$ along with (7) gives

$$A(a)\alpha b\beta A(a)+B(a,b)\alpha a\beta A(a)=0$$

Since $a\beta A(a)=0$, for all $\beta\in\Gamma$, we have $B(a,b)\alpha a\beta A(a)=0$

This implies that $A(a)\alpha b\beta A(a)=0$

By semiprimeness, we have $A(a)=0$. Thus we have

$$2T(a\gamma a)=T(a)\gamma a+a\gamma T(a) \quad (9)$$

Lemma 2.3. *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and let $T:M\to M$ be an additive mapping. Suppose that* $2T(a\alpha b\beta a)=T(a)\alpha b\beta a+a\alpha b\beta T(a)$ holds for all pairs $a,b\in M$ and $\alpha,\beta\in\Gamma$. Then

$$[T(a),a]_\alpha=0 \quad (10)$$

Proof. Replacing a by $a+b$ in relation (9)(linearization) gives

$$2T(a\gamma b+b\gamma a)=T(a)\gamma b+T(b)\gamma a+a\gamma T(b)+b\gamma T(a) \quad (11)$$

Replacing b with $2a\alpha b\beta a$ in (11) and use (1), we obtain

$$\begin{aligned}4T(&a\gamma a\alpha b\beta a+a\alpha b\beta a\gamma a)\\ &=2T(a)\gamma a\alpha b\beta a+2T(a\alpha b\beta a)\gamma a+2a\gamma T(a\alpha b\beta a)+2a\alpha b\beta a\gamma T(a)\\ &=2T(a)\gamma a\alpha b\beta a+T(a)\alpha b\beta a\gamma a+a\alpha b\beta T(a)\gamma a\\ &\quad+a\gamma T(a)\alpha b\beta a+a\gamma a\alpha b\beta T(a)+2a\alpha b\beta a\gamma T(a)\end{aligned}$$

$$\begin{aligned}4T(a\gamma a\alpha b\beta a+a\alpha b\beta a\gamma a)&=2T(a)\gamma a\alpha b\beta a+T(a)\alpha b\beta a\gamma a\\ &+a\alpha b\beta T(a)\gamma a+a\gamma T(a)\alpha b\beta a+a\gamma a\alpha b\beta T(a)+2a\alpha b\beta a\gamma T(a) \quad (12)\end{aligned}$$

Comparing (4) and (12), we arrive at

$$T(a)\alpha b\beta a\gamma a+a\gamma a\alpha b\beta T(a)-a\alpha b\beta T(a)\gamma a-a\gamma T(a)\alpha b\beta a=0 \quad (13)$$

Putting $b\gamma a$ for b in the above relation, we have

$$T(a)\alpha b\gamma a\beta a\gamma a+a\gamma a\alpha b\gamma a\beta T(a)-a\alpha b\gamma a\beta T(a)\gamma a-a\gamma T(a)\alpha b\gamma a\beta a=0 \quad (14)$$

Right multiplication of (13) by γa gives

$$T(a)\alpha b\beta a\gamma a\gamma a+a\gamma a\alpha b\beta T(a)\gamma a-a\alpha b\beta T(a)\gamma a\gamma a-a\gamma T(a)\alpha b\beta a\gamma a=0 \quad (15)$$

Subtracting (14) from (15) and using assumption (A), we get

$$a\gamma a\gamma b\beta[T(a),a]_\alpha-a\gamma b\beta[T(a),a]_\alpha\gamma a=0 \quad (16)$$

The substitution $T(a)\alpha b$ for b in (16), we have

$$a\gamma a\gamma T(a)\alpha b\beta[T(a),a]_\alpha-a\gamma T(a)\alpha b\beta[T(a),a]_\alpha\gamma a=0 \quad (17)$$

Left multiplication of (16) by $T(a)\alpha$ gives

$$T(a)\alpha a\gamma a\gamma b\beta[T(a),a]_\alpha-T(a)\alpha a\gamma b\beta[T(a),a]_\alpha\gamma a=0 \quad (18)$$

Subtracting (17) from (18), we arrive at

$$[T(a),a\gamma a]_\alpha\gamma b\beta[T(a),a]_\alpha-[T(a),a]_\alpha\gamma b\beta[T(a),a]_\alpha\gamma a=0$$

In the above relation let $x=[T(a),a\gamma a]_\alpha$, $y=[T(a),a]_\alpha$, $z=-[T(a),a]_\alpha\gamma a$ and

$c = b$. Then we have $x\gamma c\beta y + y\gamma c\beta z = 0$. Thus from Lemma 2.1, we have

$$(x+z)\gamma c\beta y = 0$$

$$\Rightarrow ([T(a), a\gamma a]_\alpha - [T(a), a]_\alpha \gamma a)\gamma b\beta [T(a), a]_\alpha = 0$$

This implies that $([T(a), a]_\alpha \gamma a + a\gamma [T(a), a]_\alpha - [T(a), a]_\alpha \gamma a)\gamma b\beta [T(a), a]_\alpha = 0$

$$\Rightarrow a\gamma [T(a), a]_\alpha \gamma b\beta [T(a), a]_\alpha = 0$$

Putting $b = b\alpha a$ in the above relation, we have

$$a\gamma [T(a), a]_\alpha \gamma b\alpha a\beta [T(a), a]_\alpha = 0$$

$$\Rightarrow a\gamma [T(a), a]_\alpha \alpha b\beta a\gamma [T(a), a]_\alpha = 0$$

using the assumption (A). By the semiprimeness of M , we obtain

$$a\gamma [T(a), a]_\alpha = 0 \tag{19}$$

Putting $a\gamma b$ for b in the relation (13), we obtain

$$T(a)\alpha a\gamma b\beta a\gamma a + a\gamma a\alpha a\gamma b\beta T(a) - a\alpha a\gamma b\beta T(a)\gamma a - a\gamma T(a)\alpha a\gamma b\beta a = 0 \tag{20}$$

Left multiplication of (13) by $a\gamma$, we have

$$a\gamma T(a)\alpha b\beta a\gamma a + a\gamma a\gamma a\alpha b\beta T(a) - a\gamma a\alpha b\beta T(a)\gamma a - a\gamma a\gamma T(a)\alpha b\beta a = 0 \tag{21}$$

Subtracting (21) from (20), and using assumption (A), we have

$$[T(a), a]_\alpha \gamma b\beta a\gamma a - a\gamma [T(a), a]_\alpha \gamma b\beta a = 0$$

Using (19) in the above relation, we obtain

$$[T(a), a]_\alpha \gamma b\beta a\gamma a = 0 \tag{22}$$

Putting $b\alpha T(a)$ for b in (22), we have

$$[T(a), a]_\alpha \gamma b\alpha T(a)\beta a\gamma a = 0 \tag{23}$$

Right multiplication of (22) by $\alpha T(a)$ gives

$$[T(a), a]_\alpha \gamma b\beta a\gamma a\alpha T(a) = 0 \tag{24}$$

Subtracting (24) from (23) and using assumption (A), we have

$$[T(a), a]_\alpha \gamma b\beta [T(a), a\gamma a]_\alpha = 0$$

The above relation can be rewritten and using (19), we have

$$[T(a), a]_\alpha \gamma b\beta [T(a), a]_\alpha \gamma a = 0$$

Putting $a\alpha b$ for b in the above relation, we obtain

$$[T(a), a]_\alpha \gamma a\alpha b\beta [T(a), a]_\alpha \gamma a = 0$$

By semiprimeness of M , we have

$$[T(a), a]_\alpha \gamma a = 0 \tag{25}$$

Replacing a by $a+b$ in (19) and then using (19) gives

$$a\gamma [T(a), b]_\alpha + a\gamma [T(b), a]_\alpha + a\gamma [T(b), b]_\alpha$$
$$+ b\gamma [T(a), a]_\alpha + b\gamma [T(a), b]_\alpha + b\gamma [T(b), a]_\alpha = 0$$

Replacing a by $-a$ in the above relation and comparing the relation so obtained with the above relation, we have

$$a\gamma[T(a),b]_\alpha + a\gamma[T(b),a]_\alpha + b\gamma[T(a),a]_\alpha = 0 \quad (26)$$

Left multiplication of (26) by $[T(a),a]_\alpha \beta$ and then use (25), we have

$$[T(a),a]_\alpha \beta b\gamma[T(a),a]_\alpha = 0$$

By semiprimeness of M , we have

$$[T(a),a]_\alpha = 0$$

Hence the relation (10) follows.

Theorem 2.1. *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and let $T : M \to M$ be an additive mapping. Suppose that $2T(a\alpha b\beta a) = T(a)\alpha b\beta a + a\alpha b\beta T(a)$ holds for all pairs $a, b \in M$ and $\alpha, \beta \in \Gamma$. Then T is a centralizer.*

Proof. The relation (9) in Lemma 2.2 and the relation (10) in Lemma 2.3 give

$$T(a\alpha a) = T(a)\alpha a \text{ and } T(a\alpha a) = a\alpha T(a)$$

since M is a 2-torsion free. Hence T is a left and also a right Jordan centralizers. By Theorem 2.1 in [7], it follows that T is a left and also a right centralizer which completes the proof of the theorem.

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