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Cordial Labelling of Cycles

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Abstract. Suppose $G=(V,E)$ be a graph with vertex set V and edge set E. A vertex labelling $f:V\to\{0,1\}$ induces an edge labelling $f^*:E\to\{0,1\}$. For $i\in\{0,1\}$, let $v_f(i)$ and $e_f(i)$ be the number of vertices v and edges e with $f(v)=i$ and $f^*(e)=i$ respectively. A graph is cordial if there exists a vertex labelling f such that $|v_f(0)-v_f(1)|\le 1$ and $|e_f(0)-e_f(1)|\le 1$.

Cycle is a closed walk in which no vertex appears more than once except beginning and ending vertices. In this paper, we label the vertices of cycles and have shown that cycles are cordial with some restrictions.

Keywords: Graph labelling; cordial labelling; cycle of finite length

AMS Mathematics Subject Classifications (2010): **05C85**

1. Introduction

In graph theory, the term cycle may refer to a closed path. If repeated vertices are allowed, it is more often called a closed walk. If a path is simple, with no repeated vertices or edges other than the starting and ending vertices, it may also be called a simple cycle, circuit, or polygon. A cycle in a directed graph is called a directed cycle. Length of any cycle is the total number of edges present in the cycle.

Let us consider a graph $G=(V,E)$ be a graph with vertex set V and edge set E. A vertex labelling $f:V\to\{0,1\}$ induces an edge labelling $f^*:E\to\{0,1\}$. For $i\in\{0,1\}$, let $v_f(i)$ and $e_f(i)$ be the number of vertices v and edges e with $f(v)=i$ and $f^*(e)=i$ respectively. A graph is cordial if there exists a vertex labelling f such that $|v_f(0)-v_f(1)|\le 1$ and $|e_f(0)-e_f(1)|\le 1$. Cordial graphs were first introduced by Cahit [1] as a weaker version of both graceful graphs and harmonious graphs. The graceful graph and harmonious graphs are discussed below.

A vertex labelling f is called a graceful labelling of a graph G with e edges if f is an injection from the vertices of G to the set $\{0,1,\ldots,e\}$ such that when each edge (x,y) is assigned the label $|f(x)-f(y)|$ the resulting edge labels are distinct. A graph G is called *graceful* if there exists a graceful labelling.

A connected graph with v vertices and $e \geq v$ edges *harmonious* if it is possible to label the vertices x with distinct elements $\lambda(x)$ of Z_e in such a way that when each edge (x,y) is labeled with $\lambda(x)+\lambda(y)$, the resulting edge labels are distinct.

In this paper, we label all the vertices of cycles of different length, joined with a common cutvertex by cordial labelling procedure and have shown that for all cases the graphs are cordial if and only if total number of edges is not congruent to 2 (mod 4).

2. Review of previous work

There are few results of cordial labelling on cycles in literature. Cordial labelling is a weaker version of graceful graphs and harmonious labellings. So we give some results of cordial, graceful and harmonious labelling on cycles which are as follows.

Ho *et al.* [8] have shown that a unicycle is cordial except for the case of C_{4k+2}. Rosa [15] have shown that the n-cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4) and Graham and Sloane [7] proved that C_n is harmonious if and only if $n \equiv$ 1 or 3 (mod 4). There are some works on cordial labelling for other graphs, which are given below.

Cahit [2] proved the followings: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the friendship graph C_3^t (i.e., the one-point union of t-cycles) if and only if $t \equiv 2$ (mod 4); for all fans are cordial; the wheel W_n is cordial if and only if $n \equiv 3$ (mod 4)(see also [6]); maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo *et al.* [12] determine all m and n for which mK_n is cordial. Liu and Zhu [13] proved that a 3-regular graph of order n is cordial if and only if $n \not\equiv 4$ (mod 8).

A k-angular cactus graph is connected graph all of whose blocks are cycles with k vertices. In [2], Cahit proved that a k-angular cactus with t cycles is cordial if $kt \not\equiv 2$ (mod 4). This was improved by Kirchherr [10] was showed any cactus whose blocks are cycles is cordial if and only if size of the graph is not congruent to 2 (mod 4). Kirchherr [11] also gave a characterization of cordial graphs in terms of their adjacency matrices. Ho *et al.* [8] showed that a unicycle is cordial unless it is

C_{4k+2} and that the generalized Petersen graph $P(n,k)$ if and only if $n \not\equiv 2$ (mod 4).

Seoud and Maqusoud [16] proved that if G is a graph with n vertices and m edges and every vertex has odd degree, then G is cordial if and only if $m+n \equiv 2$ (mod 4). They also prove the following: for $m \geq 2$, $C_n \times P_n$ is cordial except for the case $C_{4k+2} \times P_2$; P_n^4 is cordial if and only if $n \neq 4,5,6$. Seoud *et al.* [17] have proved that the following graphs are cordial: $P_n + P_m$ for all m and n except $(m,n)=(2,2)$; $C_n + C_m$ if $m \not\equiv 2$ (mod 4); $C_n + K_{1,m}$ for $m \not\equiv 3$ (mod 4) and odd m except $(n,m)=(3,1)$; $C_n + \overline{K_m}$ when n is odd, and when n is even and m is odd; $K_{1,m,n}$; the n-cube; books B_n if and only if $n \not\equiv 3$ (mod 4); $B(3,2,m)$ for all m; $B(4,3,m)$ if and only if m is even; and $B(5,3,m)$ if and only if $m \not\equiv 1$ (mod 4).

Diab [4] has proved that if G and H are cordial and one has even size, then $G \cup H$ is cordial; if G and H are cordial and both have even size, then $G+H$ is cordial; if G and H are cordial and one has even size and either one has even order, then $G+H$ is cordial; $C_n \cup C_m$ if and only if $m+n \not\equiv 2$ (mod 4); mC_n if and only if $(m,n) \neq (3,3)$ and $\{m \,(\mathrm{m}od\ 4), n\,(\mathrm{m}od\ 4)\} \neq \{0,2\}$; and if P_n^k is cordial, then $n \geq k+1+\sqrt{k-1}$.

Shee and Ho [18] have investigated the cordiality of the one-point union of n copies of various graphs.

Du [5] proved that the disjoint union of $n \geq 2$ wheels is cordial if and only if n is even or n is odd and the number of vertices of in each cycle is not 0 (mod 4), or n is odd and the number of vertices in each cycle is not 3 (mod 4). Hovey [9] has obtained the following: caterpillars are k-cordial; all trees are k-cordial for $k=3,4$ and 5; odd cycles with pendent edges attached are k-cordial for all k; cycles are k-cordial for all odd k; for k even C_{2mk+j} is cordial when $0 \leq j \leq \dfrac{k}{2}+2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not k-cordial; and for k even, K_{mk} is k-cordial if and only if $m=1$.

Cairnie and Edwards [3] have determined the computation complexity of cordial and k-cordial labellings. They proved the conjecture Kirchherr [11] that deciding whether a graph admits a cordial labelling if NP-complete.

In [14], Ramanjaneyulu proved Pl_n, $n \geq 5$ is cordial if $n \not\equiv 0$ (mod 4); $Pl_{m,n}$, $m,n \geq 3$ is cordial; $Pl_{m,n}$ is total product cordial except for either m even and $n \not\equiv 2$ (mod 4), or m odd and $n \not\equiv 1$ (mod 4).

3. Cordial labelling of cycles

We label all the vertices of C_n, $n=1,2,3,4$ having a common cutvertex by using cordial labelling and the results are discussed below. Suppose P_n is a path with n vertices. If we merge first and last vertices of P_n, we obtain the cycle C_{n-1}, i.e., a cycle of length $n-1$. Now the cordial labelling of path and cycles are discussed below. Ho *et al.* [8] proved that a unicycle is cordial except it is C_{4k+2}. We discuss this in the following lemma.

Lemma 1. *A path* $P_n(\geq 3)$ *of length* $n-1$ *is cordial.*

Proof. Let the vertices and edges of P_n be $v_0, v_1, v_2, \ldots, v_{n-1}$; $e_0, e_1, e_2, \ldots, e_{n-2}$ respectively, where $e_i = (v_i, v_{i+1})$, $i = 0,1,\ldots,n-2$. Now we label the vertices as in the following way.

$$f(v_i)=\begin{cases}0, \text{ if } i\equiv 0 \pmod 4;\\ 0, \text{ if } i\equiv 1 \pmod 4;\\ 1, \text{ if } i\equiv 2 \pmod 4;\\ 1, \text{ if } i\equiv 3 \pmod 4, \ i=0,1,\ldots,n-1.\end{cases}$$

Now if n is even then $v_f(0)=\dfrac{n}{2}=v_f(1)$ and $e_f(0)=\dfrac{n}{2}$, $e_f(1)=\dfrac{n-2}{2}$.

But if n is odd then $v_f(0)=\dfrac{n+1}{2}$, $v_f(1)=\dfrac{n-1}{2}$ and $e_f(0)=\dfrac{n-1}{2}=e_f(1)$.

Thus, $|v_f(0)-v_f(1)|=0$, $|e_f(0)-e_f(1)|=1$ if n is even and $|v_f(0)-v_f(1)|=1$, $|e_f(0)-e_f(1)|=0$ if n is odd. So, P_n is cordial.

Lemma 2. *[8] A cycle* C_n *of length n is cordial.*

Proof. Let the vertices and edges of C_n be $v_0, v_1, v_2, \ldots, v_{n-1}$; $e_0, e_1, e_2, \ldots, e_{n-1}$ respectively, where $e_i = (v_i, v_{i+1})$, $i=0,1,\ldots,n-2$ and $e_{n-1}=(v_0, v_{n-1})$. To label the vertices of C_n we classify the cycle into four groups, viz., C_{4k}, C_{4k+1}, C_{4k+2} and C_{4k+3}.

Case 1. <u>If $n = 4k \equiv 0 \pmod 4$</u>.

In this case, $f(v_i)$ is defined as follows.

$$f(v_i)=\begin{cases}0, \text{ if } i\equiv 0 \pmod 4;\\ 0, \text{ if } i\equiv 1 \pmod 4;\\ 1, \text{ if } i\equiv 2 \pmod 4;\\ 1, \text{ if } i\equiv 3 \pmod 4, \ i=0,1,\ldots,n-1.\end{cases}$$

Here, $v_f(0)=\dfrac{n}{2}=2k=v_f(1)$ and $e_f(0)=\dfrac{n}{2}=2k=e_f(1)$, for $k=1,2,\ldots,\dfrac{n}{4}$. That is, $|v_f(0)-v_f(1)|=0$ and $|e_f(0)-e_f(1)|=0$. Thus C_{4k} is cordial.

Case 2. <u>If $n=4k+1\equiv 1 \pmod 4$</u>.

Here we label the first $n-1=4k$ vertices according to the same rule as in the above case. For the last vertex, f is defined as $f(v_{4k})=1$.

Here, $v_f(0)=\dfrac{n-1}{2}=\dfrac{4k+1-1}{2}=2k$, $v_f(1)=\dfrac{n+1}{2}=\dfrac{4k+1+1}{2}=2k+1$.

And $e_f(0)=\dfrac{n+1}{2}=\dfrac{4k+1+1}{2}=2k+1$, $e_f(1)=\dfrac{n-1}{2}=\dfrac{4k+1-1}{2}=2k$, for $k=1,2,\ldots,\dfrac{n-1}{4}$. Thus, $|v_f(0)-v_f(1)|=1$ and $|e_f(0)-e_f(1)|=1$.

Hence C_{4k+1} is cordial.

Case 3. <u>If $n=4k+2\equiv 2 \pmod 4$</u>.

Here also we label first $n-2=4k$ vertices as the same process as given in case 1. For the last two vertices v_{4k} and v_{4k+1}, f is defined as

$f(v_{4k})=0$ and $f(v_{4k+1})=1$ respectively.

Here, $v_f(0)=\dfrac{n}{2}=2k+1=v_f(1)$ and $e_f(0)=\dfrac{n-2}{2}=2k$ and $e_f(1)=\dfrac{n+2}{2}=2k+2$, for $k=1,2,\ldots,\dfrac{n-2}{4}$. In this case, $|v_f(0)-v_f(1)|=0$ but $|e_f(0)-e_f(1)|=2\not< 1$.Thus C_{4k+2} is not cordial.

Case 4. <u>If $n=4k+3\equiv 3 \pmod 4$</u>.

The labeling of the vertices v_i's; $i=0,1,\ldots,4k+1$, are same as given in case 1. Then we label the last vertex v_{4k+2} as $f(v_{4k+2})=1$.

Here, $v_f(0)=\dfrac{n+1}{2}=2k+2$, $v_f(1)=\dfrac{n-1}{2}=2k+1$ and

$e_f(0)=\dfrac{n-1}{2}=2k+1$, $e_f(1)=\dfrac{n+1}{2}=2k+2$, for $k=1,2,\ldots,\dfrac{n-3}{4}$. Here $|v_f(0)-v_f(1)|=1$ and $|e_f(0)-e_f(1)|=1$. Hence C_{4k+3} is cordial.

Hence cycle is cordial except of length $4k+2$.

The different cases of cordial labelling of unicycle are shown in Table 1.

Values of n	Condition on vertex	Condition on edge	Cordial
$n=4k$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)$	Yes
$n=4k+1$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)+1$	Yes
$n=4k+2$	$v_f(0)=v_f(1)$	$e_f(0)+2=e_f(1)$	No
$n=4k+3$	$v_f(0)=v_f(1)+1$	$e_f(0)+1=e_f(1)$	Yes

Table 1: Vertex and edge conditions of Cases 1, 2, 3 and 4 of Lemma 2

Lemma 3. *Let a graph G contains two cycles of lengths n and m* $(n,m>3)$ *respectively and they are joined by a common cutvertex. Then G is cordial if and only if* $n+m\not\equiv 2$ *(mod 4).*

Proof. Let v_i, e_i; $i=0,1,\ldots,n-1$ be the vertices and edges of C_n and v_0, v'_1, $v'_2,\ldots,v'_{m-1}$; e'_0, $e'_1,\ldots$, e'_{m-1} be that of C_m. They are joined with the cutvertex v_0, where $e_i=(v_i,v_{i+1})$, for $i=0,1,\ldots,n-2$, $e_{n-1}=(v_0,v_{n-1})$; $e'_j=(v'_j,v'_{j+1})$, $j=1,2,\ldots,m-2$, $e'_0=(v_0,v'_1)$ and $e'_{m-1}=(v_0,v'_{m-1})$ respectively.

Now we label the vertices of the graph G by cordial labelling. The different cases are discussed below.

Case 1. <u>If $n=4k$, $m=4k+i$, for $i=0,1,2,3$</u>.

We label the vertices of C_n according to the rule as in case 1 in Lemma 2. Now the label of C_m are done by the following rule.

Case 1.1. <u>For $m=4k$</u>.

For the vertices v'_j; $j=4,5,\ldots,m-1$,

$$f(v'_j)=\begin{cases}1, & \text{if } j\equiv 0 \pmod 4;\\ 0, & \text{if } j\equiv 1 \pmod 4;\\ 0, & \text{if } j\equiv 2 \pmod 4;\\ 1, & \text{if } j\equiv 3 \pmod 4;\end{cases}$$

and for the other vertices, $f(v'_j)=0$, for $j=1,2$; and $f(v'_3)=1$.

In this case, the total number of vertices $=n+m-1=8k-1$ and edges $=n+m=8k$, where k is a positive integer.

Then, $v_f(0)=\dfrac{8k-1+1}{2}=4k$, $v_f(1)=\dfrac{8k-1-1}{2}=4k-1$ and $e_f(0)=\dfrac{8k}{2}=4k=e_f(1)$. Here $|v_f(0)-v_f(1)|=1$ and $|e_f(0)-e_f(1)|=0$.

Case 1.2. For $m=4k+1$.

We label v'_j; $j=1,2,\ldots,m-2=4k-1$, as same process of the labelling of C_m as given in case 1.1. Then we label v_{4k} as $f(v'_{4k})=1$.

Now we get, $v_f(0)=\dfrac{8k}{2}=4k=v_f(1)$ and $e_f(0)=\dfrac{8k+1+1}{2}=4k+1$, $e_f(1)=\dfrac{8k+1-1}{2}=4k$.

Case 1.3. For $m=4k+2$.

The labelling of first $m-2=4k+1$ vertices of C_m are same as given in above case except of the labelling of v_{4k+1}. We label v_{4k+1} as $f(v'_{4k+1})=1$.

Here, $v_f(0)=\dfrac{8k+1-1}{2}=4k$, $v_f(1)=\dfrac{8k+1+1}{2}=4k+1$ and $e_f(0)=\dfrac{8k+2+2}{2}=4k+2$, $e_f(1)=\dfrac{8k+2-2}{2}=4k$.

Case 1.4. For $m=4k+3$.

The labelling of the vertices v'_j; $j=4,5,\ldots,m-2$ as

$$f(v'_j)=\begin{cases}0, & \text{if } j\equiv 0 \pmod 4;\\ 0, & \text{if } j\equiv 1 \pmod 4;\\ 1, & \text{if } j\equiv 2 \pmod 4;\\ 1, & \text{if } j\equiv 3 \pmod 4.\end{cases}$$

Then we label other vertices as

$f(v'_1)=0$; $f(v'_j)=1$, for $j=2,3$ and $m-1$.

Here, $v_f(0)=\dfrac{8k+2}{2}=4k+1=v_f(1)$ and $e_f(0)=\dfrac{8k+3-1}{2}=4k+1$, $e_f(1)=\dfrac{8k+3+1}{2}=4k+2$. The summary of case 1 are cited in Table 2.

Values of n,m	Condition on vertex	Condition on edge	Cordial
$n=4k$, $m=4k$	$v_f(0)=v_f(1)+1$	$e_f(0)=e_f(1)$	Yes
$n=4k$, $m=4k+1$	$v_f(0)=v_f(1)$	$e_f(0)=e_f(1)+1$	Yes
$n=4k$, $m=4k+2$	$v_f(0)+1=v_f(1)$	$e_f(0)=e_f(1)+2$	No
$n=4k$, $m=4k+3$	$v_f(0)=v_f(1)$	$e_f(0)+1=e_f(1)$	Yes

Table 2: Vertex and edge conditions table of Case 1 of Lemma 3

Case 2. If $n=4k+1$, $m=4k+i$, for $i=1,2,3$.

In this case, we first label the vertices of C_n according to the rule given in case 2 of Lemma 2 except the case 2.2. Then we label C_m in the following ways.

Case 2.1. For $m=4k+1$.

For this case we label the vertices of C_m as the same process as in case 1.2 of this lemma.

Now we get, $v_f(0)=\dfrac{n+m-1}{2}=4k$, $v_f(1)=\dfrac{8k+1+1}{2}=4k+1$ and $e_f(0)=\dfrac{8k+2+2}{2}=4k+2$, $e_f(1)=\dfrac{8k+2-2}{2}=4k$.

Case 2.2. For $m=4k+2$.

First we label C_m according to the rule given in case 2 (for $n=4k+2$) of Lemma 2. Then we label the cycle C_n as the same process as the labelling of C_m (for $m=4k+1$) given in case 1.2.

Here, $v_f(0)=\dfrac{8k+2}{2}=4k+1=v_f(1)$ and $e_f(0)=\dfrac{8k+3-1}{2}=4k+1$, $e_f(1)=\dfrac{8k+3+1}{2}=4k+2$.

Case 2.3. For $m=4k+3$.

The labelling of C_m is same as given in case 1.4 of this lemma.

Here, $v_f(0)=\dfrac{8k+3-1}{2}=4k+1$, $v_f(1)=\dfrac{8k+3+1}{2}=4k+2$ and $e_f(0)=\dfrac{8k+4}{2}=4k+2=e_f(1)$.

Values of n, m	Condition on vertex	Condition on edge	Cordial
$n = 4k+1$, $m = 4k+1$	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)+2$	No
$n = 4k+1$, $m = 4k+2$	$v_f(0) = v_f(1)$	$e_f(0)+1 = e_f(1)$	Yes
$n = 4k+1$, $m = 4k+3$	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)$	Yes

Table 3: Vertex and edge conditions table of Case 2 of Lemma 3

Case 3. If $n = 4k+2$, $m = 4k+2$.

First we label the vertices of C_n as given in case 3 of previous lemma. Then we label C_m as the same rule as in case 1.3.

From this case, we get, $v_f(0) = \dfrac{8k+3+1}{2} = 4k+2$, $v_f(1) = \dfrac{8k+3-1}{2} = 4k+1$ and $e_f(0) = \dfrac{8k+4}{2} = 4k+2 = e_f(1)$.

Case 4. If $n = 4k+2$, $m = 4k+3$.

In this case, we first label the vertices of C_m according to the rule given in case 4 (for $n = 4k+3$) of Lemma 2. Now we label C_n as the same process of labelling of C_m given in case 1.3.

Here, $v_f(0) = \dfrac{8k+4}{2} = 4k+2 = v_f(1)$ and $e_f(0) = \dfrac{8k+5+1}{2} = 4k+3$, $e_f(1) = \dfrac{8k+5-1}{2} = 4k+2$.

Case 5. If $n = 4k+3$, $m = 4k+3$.

The labelling procedure of C_n is same as given in case 4 of previous lemma. And C_m as in case 1.4 of this lemma.

Here, $v_f(0) = \dfrac{8k+5+1}{2} = 4k+3$, $v_f(1) = \dfrac{8k+5-1}{2} = 4k+2$ and $e_f(0) = \dfrac{8k+6-2}{2} = 4k+2$, $e_f(1) = \dfrac{8k+6+2}{2} = 4k+4$.

Values of n, m	Condition on vertex	Condition on edge	Cordial
$n = 4k+2$, $m = 4k+2$	$v_f(0) = v_f(1)+1$	$e_f(0) = e_f(1)$	Yes
$n = 4k+2$, $m = 4k+3$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)+1$	Yes

$n = 4k+3$, $m = 4k+3$	$v_f(0) = v_f(1)+1$	$e_f(0)+2 = e_f(1)$	No

Table 4: Vertex and edge conditions table of Cases 3, 4 and 5 of Lemma 3

From all above cases, we see that the graph having two cycles is cordial except three cases. The cases are $n = 4k$, $m = 4k+2$; $n = 4k+1$, $m = 4k+1$; and $n = 4k+3$, $m = 4k+3$ respectively. In all these cases, $m+n \equiv 2$ (mod 4). Thus, G is cordial if and only if the number of edges is not congruent to 2 (mod 4).

Lemma 4. *If a graph contains three cycles of lengths n,m and p* $(n,m,p > 3)$, *joined with a common cutvertex, then it is cordial.*

Proof. Let v_i, e_i; $i = 0,1,\ldots,n-1$ be the vertices and edges of C_n; v_0, v'_1, $v'_2,\ldots,v'_{m-1}$; e'_0, $e'_1,\ldots,$ e'_{m-1} that of C_m; and v_0 v''_1, $v''_2,\ldots,v''_{p-1}$; e''_0, $e''_1,\ldots,e''_{p-1}$ that of C_p respectively. They are joined with the cutvertex v_0, where $e_i = (v_i, v_{i+1})$, for $i = 0,1,\ldots,n-2$, $e_{n-1} = (v_0, v_{n-1})$; $e'_j = (v'_j, v'_{j+1})$, $j = 1,2,\ldots,m-2$, $e'_0 = (v_0, v'_1)$ and $e'_{m-1} = (v_0, v'_{m-1})$; $e''_j = (v'_{k_1}, v'_{k_1+1})$, $k_1 = 1,2,\ldots,p-2$, $e''_0 = (v_0, v''_1)$ and $e''_{p-1} = (v_0, v''_{p-1})$ respectively.

Now we label the vertices according to the following rule.

Case 1. <u>If $n = 4k$, $m = 4k$, $p = 4k+i$, for $i = 0,1,2,3$</u>.

First we label the cycles C_n and C_m as the same procedure as in case 1.1 of previous lemma. Now we label the other cycles.

Case 1.1. <u>For $p = 4k$</u>.

For v''_{k_1}; $k_1 = 4,5,\ldots,p-1$,

$$f(v''_{k_1}) = \begin{cases} 0, \text{ if } k_1 \equiv 0 \text{ (mod 4)}; \\ 0, \text{ if } k_1 \equiv 1 \text{ (mod 4)}; \\ 1, \text{ if } k_1 \equiv 2 \text{ (mod 4)}; \\ 1, \text{ if } k_1 \equiv 3 \text{ (mod 4)}; \end{cases}$$

then $f(v''_1) = 0$, $f(v''_{k_1}) = 1$, for $k_1 = 2,3$.

Here, $v_f(0) = \dfrac{12k-2}{2} = 6k-1 = v_f(1)$ and $e_f(0) = \dfrac{12k}{2} = 6k = e_f(1)$.

Case 1.2. <u>For $p = 4k+1$</u>.

Here we label the vertices of C_p as same process as given in above case except for the vertex v''_{p-1}. And we define f of that vertex as $f(v''_{p-1}) = 1$.

Here, $v_f(0) = \frac{12k-1-1}{2} = 6k-1$, $v_f(1) = \frac{12k-1+1}{2} = 6k$ and $e_f(0) = \frac{12k+1+1}{2} = 6k+1$, $e_f(1) = \frac{12k+1-1}{2} = 6k$.

Case 1.3. <u>For $p = 4k+2$.</u>

We label the vertices v''_4, v''_5, …, v''_{p-3} as the same process as in case 1.1 of this lemma. Now we label the other vertices as

$$f(v''_{k_1}) = \begin{cases} 0, \text{ if } k_1 = 1, p-2; \\ 1, \text{ if } k_1 = 2, p-1. \end{cases}$$

In this case, $v_f(0) = \frac{12k}{2} = 6k = v_f(1)$ and $e_f(0) = \frac{12k+2-2}{2} = 6k$, $e_f(1) = \frac{12k+2+2}{2} = 6k+2$.

Case 1.4. <u>For $p = 4k+3$.</u>

The labelling of the vertices v''_4, v''_5, …, v''_{p-4} are same as in case 1.1 of this lemma. For other vertices f is defined as

$$f(v''_{k_1}) = \begin{cases} 0, \text{ if } k_1 = 1, p-3, p-2; \\ 1, \text{ if } k_1 = 2, 3, p-1. \end{cases}$$

Here, $v_f(0) = \frac{12k+1+1}{2} = 6k+1$, $v_f(1) = \frac{12k+1-1}{2} = 6k$ and $e_f(0) = \frac{12k+3-1}{2} = 6k+1$, $e_f(1) = \frac{12k+3+1}{2} = 6k+2$.

Values of n, m, p	Condition on vertex	Condition on edge	Cordial
$n = 4k$, $m = 4k$, $p = 4k$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$	Yes
$n = 4k$, $m = 4k$, $p = 4k+1$	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)+1$	Yes
$n = 4k$, $m = 4k$, $p = 4k+2$	$v_f(0) = v_f(1)$	$e_f(0)+2 = e_f(1)$	No
$n = 4k$, $m = 4k$, $p = 4k+3$	$v_f(0) = v_f(1)+1$	$e_f(0)+1 = e_f(1)$	Yes

Table 5: Vertex and edge conditions table of Case 1 of Lemma 4

All other possible cases are summarized in Table 6.

Values of n, m, p	Condition on vertex	Condition on edge	Cordial
$n = 4k$, $m = 4k+1$, $p = 4k+1$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1) + 2$	No
$n = 4k$, $m = 4k+1$, $p = 4k+2$	$v_f(0)+1 = v_f(1)$	$e_f(0)+1 = e_f(1)$	Yes
$n = 4k$, $m = 4k+1$, $p = 4k+3$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$	Yes
$n = 4k$, $m = 4k+2$, $p = 4k+2$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$	Yes
$n = 4k$, $m = 4k+2$, $p = 4k+3$	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)+1$	Yes
$n = 4k$, $m = 4k+3$, $p = 4k+3$	$v_f(0) = v_f(1)$	$e_f(0)+2 = e_f(1)$	No
$n = 4k+1$, $m = 4k+1$, $p = 4k+1$	$v_f(0)+1 = v_f(1)$	$e_f(0)+1 = e_f(1)$	Yes
$n = 4k+1$, $m = 4k+1$, $p = 4k+2$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$	Yes
$n = 4k+1$, $m = 4k+1$, $p = 4k+3$	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)+1$	Yes
$n = 4k+1$, $m = 4k+2$, $p = 4k+2$	$v_f(0)+1 = v_f(1)$	$e_f(0) = e_f(1)+1$	Yes
$n = 4k+1$, $m = 4k+2$, $p = 4k+3$	$v_f(0) = v_f(1)$	$e_f(0)+2 = e_f(1)$	No
$n = 4k+1$, $m = 4k+3$, $p = 4k+3$	$v_f(0) = v_f(1)+1$	$e_f(0)+1 = e_f(1)$	Yes
$n = 4k+2$, $m = 4k+2$, $p = 4k+2$	$v_f(0)+2 = v_f(1)$	$e_f(0)+2 = e_f(1)$	No
$n = 4k+2$, $m = 4k+2$, $p = 4k+3$	$v_f(0)+1 = v_f(1)$	$e_f(0)+1 = e_f(1)$	Yes
$n = 4k+2$, $m = 4k+3$, $p = 4k+3$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$	Yes
$n = 4k+3$, $m = 4k+3$, $p = 4k+3$	$v_f(0) = v_f(1)+1$	$e_f(0) = e_f(1)+1$	Yes

Table 6: Vertex and edge conditions table of all other cases of Lemma 4

Therefore, from all the above cases we see that the graph G is cordial if and only if the total number of edges is not congruent to 2 (mod 4).

The lemma 4 can be extended for four cycles stated below.

Lemma 5. *If a graph contains four cycles of lengths n,m, p and q* $(n, m, p, q > 3)$

respectively, joined with a common cutvertex, is cordial if and only if $n+m+p+q \not\equiv 2$ *(mod 4).*

From all the above cases we see that the graph is not cordial when total number of edges is congruent to 2 (mod 4). Thus, the graph is cordial if and only if total number of edges is not congruent to 2 (mod 4).

These result can be extended for any number of cycles stated below.

Theorem 1. *Let G be a graph contains finite number of cycles of finite lengths, joined with a common cutvertex. Then the graph is cordial if and only if total number of edges of G is not congruent to 2 (mod 4).*

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