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b-coloring in Square of Cartesian Product of Two Cycles

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Abstract. A b-coloring of a graph G with k colors is a proper coloring of G using k colors in which each color class contains a color dominating vertex, that is, a vertex which has at least one neighbor in each of the other color classes. The largest integer k(>0) for which G has a b-coloring using k colors is the b-chromatic number b(G) of G. In this paper, we obtain the b-chromatic number of the square of Cartesian product of two cycles. Further, we obtained the b-coloring number of $C_n^k \square K_2$ and $C_n^k \square K_3$ and prove that these graphs are b-continuous for some particular values of n.

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Keywords: Cartesian product, product of a cycle, b-coloring, b-continuous

1. Introduction

All graphs in this paper are finite, simple and undirected graphs. A k-vertex coloring of a graph G is an assignment of k colors 1, 2, ..., k, to the vertices. The coloring is proper if no two distinct adjacent vertices share the same color. A graph G is k-colorable if G has a proper k-vertex coloring. The chromatic number $\chi(G)$ is the minimum number k such that G is k-colorable. Color of a vertex v is denoted by c(v).

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes. In other words, each color class contains a color dominating vertex (a vertex which has a neighbor in all the other color classes). The b-chromatic number b(G) is the largest integer k such that G admits a b-coloring with k colors. The b-spectrum $S_b(G)$ of G is defined by

$S_b(G) = \{ k \in N : \chi(G) \leq k \leq b(G)$ and G is b-colorable with k colors $\}$. It is known that $\chi(G) \leq b(G) \leq \Delta+1$.

A graph G is b-continuous if $S_b(G) = [\chi(G), b(G)]$. R.W. Irving and D.F. Manlove [5] have shown that the problem of determining the b-chromatic number is NP-hard for general graphs, but polynomial-time solvable for trees. Also they proved that some graphs admits b-coloring but not b-continuous. Further they proved that the 3-dimensional cube Q_3 is not b-continuous [5]. T. Faik proved that some classes of graphs are known to be b-continuous [3].

A graph G_1 is called covering of G with projection $f : G_1 \rightarrow G$ if there is a surjection $f:V(G_1) \rightarrow V(G)$ such that $f|N(v_1) : N(v_1) \rightarrow N(v)$ is a bijection for any vertex $v \in V(G)$ and $v_1 \in f^{-1}(v)$ [6].

The Cartesian product $G \square H$ of two graphs G and H, is the graph with vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H) = \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E(G)$ with $y_1=y_2$ or $(y_1, y_2) \in E(H)$ with $x_1=x_2 \}$. [4]. The square G^2 of a graph G is defined on the vertex set of G in such a way that distinct vertices with distance at most 2 in G are joined by an edge.

In this section, we obtain the b-chromatic number of the square of Cartesian product $C_m \square C_n$ of two cycles when m and n are multiples of 13. A graph is a power of cycle, denoted C_n^k, if $V(C_n^k) = \{v_0 (= v_n), v_1, v_2, \ldots, v_{n-1}\}$ and $E(C_n^k) = E_1 \cup E_2 \cup \ldots \cup E_k$, where $E_i=\{(v_j, v_{(j+i)}(\text{mod } n)) : 0 \leq j \leq n-1\}$ [5]. Note that C_n^k is a 2k-regular graph and that $k \geq 1$.

In this paper we study the b-chromatic number of the square of Cartesian product $C_m \square C_n$ of two cycles when m and n are multiples of 13. In such cases, we give the color classes.

2. Square of Cartesian product of two cycles

In this section, we prove that the square of the graph $C_m \square C_n$ is b-continuous when m and n are integers multiples of 13.

Lemma 2.1 Let G be the square of the graph $C_{13} \square C_{13}$ Then G is b- colorable with 13- colors and b(G) = 13.

Proof. Let the vertex set of G be $V=\{(i,j) : 1 \leq i \leq 13, 1 \leq j \leq 13\}$. Since $b(G) = \Delta+1$ and $\Delta(G) = 12$, we have b(G) = 13. It remains to show that G is b-colorable with 13 colors. Let us color the vertices of G us follows:

c((1, 1)) = 4, c((1, 2)) = 11, c((1, 3)) = 2, c((1, 4)) = 10, c((1, 5)) = 12, c((1, 6)) = 5, c((1, 7)) = 13, c((1,8)) = 3, c((1, 9)) = 8, c((1, 10)) = 9, c((1, 11)) = 1, c((1, 12)) = 6, c((1,13)) = 7,

c((2, 1)) = 5, c((2, 2)) = 13, c((2, 3)) = 3, c((2, 4)) = 8, c((2, 5)) = 9, c((2, 6)) = 1, c((2, 7)) = 6, c((2,8)) = 7, c((2,9)) = 4, c((2,10)) = 11, c((2, 11)) = 2, c((2, 12)) = 10, c((2, 13)) = 12,

c((3, 1)) = 1, c((3, 2)) = 6, c((3, 3)) = 7, c((3, 4)) = 4, c((3, 5)) = 11, c((3, 6)) = 2, c((3,7)) = 10, c((3,8)) = 12, c((3,9)) = 5, c((3,10)) = 13, c((3,11)) = 3, c((3, 12)) = 8, c((3, 13)) = 9,

c((4, 1)) = 2, c((4, 2)) = 10, c((4, 3)) = 12, c((4, 4)) = 5, c((4, 5)) = 13, c((4, 6)) = 3, c((4, 7)) = 8, c((4, 8)) = 9, c((4, 9)) = 1, c((4, 10)) = 6, c((4, 11)) = 7, c((4, 12)) = 4, c((4, 13)) = 11,
c((5, 1)) = 3, c((5, 2)) = 8, c((5, 3)) = 9, c((5, 4)) = 1, c((5, 5)) = 6, c((5, 6)) = 7, c((5, 7)) = 4, c((5,8)) = 11, c((5,9)) = 2, c((5,10)) = 10, c((5,11)) = 12, c((5,12)) = 5, c((5, 13)) = 13,
c((6, 1)) = 7, c((6, 2)) = 4, c((6, 3)) = 11, c((6, 4)) = 2, c((6, 5)) = 10, c((6, 6)) = 12, c((6, 7)) = 5, c((6, 8)) = 13, c((6, 9)) = 3, c((6, 10)) = 8, c((6,11)) = 9, c((6, 12)) = 1, c((6, 13)) = 6,
c((7, 1)) = 12, c((7,2)) = 5, c((7, 3)) = 13, c((7, 4)) = 3, c((7, 5)) = 8, c((7, 6)) = 9, c((7, 7)) = 1, c((7, 8)) = 6, c((7, 9)) = 7, c((7,10)) = 4, c((7, 11)) = 11, c((7,12)) = 2, c((7, 13)) = 10,
c((8, 1)) = 9, c((8, 2)) = 1, c((8, 3)) = 6, c((8, 4)) = 7, c((8, 5)) = 4, c((8, 6)) = 11, c((8,7)) = 2, c((8,8)) = 10, c((8,9)) = 12, c((8,10)) = 5, c((8,11)) = 13, c((8, 12)) = 3, c((8, 13)) = 8,
c((9, 1)) = 11, c((9,2)) = 2, c((9, 3)) = 10, c((9, 4)) = 12, c((9, 5)) = 5, c((9, 6)) = 13, c((9, 7)) = 3, c((9, 8)) = 8, c((9, 9)) = 9, c((9, 10)) = 1, c((9, 11)) = 6, c((9, 12)) = 7, c((9, 13)) = 4,
c((10,1)) = 13, c((10,2)) = 3, c((10,3)) = 8, c((10,4)) = 9, c((10,5)) = 1, c((10,6))= 6, c((10, 7)) = 7, c((10, 8)) = 4, c((10, 9)) = 11, c((10, 10)) = 2, c((10, 11)) = 10, c((10, 12)) = 12, c((10, 13)) = 5,
c((11,1)) = 6, c((11,2)) = 7, c((11,3)) = 4, c((11,4)) = 11, c((11,5)) = 2, c((11, 6))= 10, c((11, 7)) = 12, c((11, 8)) = 5, c((11, 9)) = 13, c((11, 10)) = 3, c((11, 11)) = 8, c((11, 12)) = 9, c((11, 13)) = 1,
c((12, 1)) = 10, c((12, 2)) = 12, c((12, 3)) = 5, c((12, 4)) = 13, c((12, 5)) = 3, c((12, 6)) = 8, c((12, 7)) = 9, c((12,8)) = 1, c((12, 9)) = 6, c((12, 10)) = 7, c((12, 11)) = 4, c((12,12)) = 11, c((12, 13)) = 2,
c((13, 1)) = 8, c((13, 2)) = 9, c((13, 3)) = 1, c((13, 4)) = 6, c((13, 5)) = 7, c((13, 6)) = 4, c((13, 7)) = 11, c((13, 8)) = 2, c((13, 9)) = 10, c((13, 10)) = 12, c((13,11)) = 5, c((13, 12)) = 13, c((13, 13)) = 3.

Note that in the above coloring all the vertices are colorful. □

In [1], S. Chandra Kumar and T. Nicholas proved the following theorem.

Lemma 2.2. [1] Let $f : G \rightarrow H$ be a covering projection from a graph G on to another graph H. If the graph H is b-colorable with k colors, then so is G.

Theorem 2.3. Let m and n be integers multiples of 13 and G be the square of the graph $C_m \square C_n$. Then G is b-colorable with 13 colors and b(G) = 13.

Proof. Let V(G) = {(x, y) : $1 \le x \le m$, $1 \le y \le n$} be the vertex set of G. Let H be the square of $C_{13} \square C_{13}$.

Define $f : G \rightarrow H$ by $f((x, y)) = (x(\text{mod } 13), y(\text{mod } 13))$. Then f is a covering projection from G onto H. The rest of the proof follows from Lemma 2.1 and Lemma 2.2. □

3. b-coloring of $C_n^k \square K_2$ and $C_n^k \square K_3$

In this section, by using the covering projection, we prove that the graphs $C_n^k \square K_2$ and $C_n^k \square K_3$ are b-continuous for some values of n.

Lemma 3.1. If $2k + 2$ divides n, then $b(G) = 2k + 2$, where $G = C_n^k \square K_2$.

Proof. Consider the graph $H = C_{2k+2}^k \square K_2$. First we show that $b(H) = 2k + 2$. Let $V(C_{2k+2}^k) = \{0(= 2k + 2),1, 2, . . . ,2k + 1\}$ be the vertex set of C_{2k+2}^k and $\{0,1\}$ be the vertex set of K_2. Note that, for any vertex $(a, b)\in V(H)$, $|N((a, b))| = 2k + 1$ and $N ((a, b)) = \{(a, b\oplus_2 1), (a \oplus_h 1, b), (a\oplus_h 2, b), . . ., (a\oplus_h k, b), (a \oplus_h (h-1), b), (a\oplus_h (h-2), b),. . . , (a\oplus_h (h-k), b)\}$, where $\oplus_h$ is the operation, addition modulo $h = 2k+2$.

Since $\Delta(H) = 2k + 1$, $b(H) \leq 2k + 2$. It remains to show that H is b-colorable with $2k + 2$ colors. Let us color the vertices of H with $2k + 2$ colors 1, 2,. . . , $2k + 2$ as follows:
$c((1,0)) = 1$, $c((2,0)) = 2$, …, $c((2k + 2,0)) = c((0,0)) = 2k+2$ and $c((k+1)+1,1) = 1$, $c((k+1)+2,1) = 2$, …, $c((k+1)+k, k) = c(2k +1, k) = k$, $c(0,1) = k + 1$, $c(1,1) = k + 2$, $c(2, 1) = k + 3$, . .., $c(k + 1,1) = 2k + 2$. Note that, in the above coloring, each vertex is colorful.

Consider the graph $G = C_n^k \square K_2$. Define $f : G\rightarrow H$ by $f((a, b)) = (a \text{ mod } (2k + 2),\ b \text{ mod } 2)$. Since $2k + 2$ divides n, the function f is a covering projection from G onto H. Since H is b-colorable with $2k + 2$ colors, by Lemma 2.2, G is also b-colorable with $2k + 2$ colors.

Since $\Delta(G) = 2k + 1$, $b(G) = 2k + 2$. □

Lemma 3.2. A simple connected graph G with at least 2 vertices is b-colorable with d colors, then so is $G\square K_2$.

Proof. Let G be a b-colorable graph with d colors. Let the corresponding color classes be $C_1, C_2, …, C_d$ with colors 1, 2,. . . , d respectively. Let $v_1, v_2, …, v_d$ be the colorful vertices of colors 1, 2, …, d respectively. Let us color the vertices of $G\square K_2$ as follows:
$c((v, 0)) = i$ if $v\in C_i$ for all $1 \leq i \leq d$, $c((v, 1)) = i + 1$ if $v\in C_i$ for all $1\leq i \leq d-1$ and $c((v, 1)) = 1$ if $v\in C_d$. From the above coloring, it is easy to observe that the vertices $(v_1, 0), (v_2, 0), …, (v_d, 0)$ are colorful vertices of colors 1, 2,. . . , d respectively. □

In [2], S. Chandra Kumar and T. Nicholas have proved the following lemma.

Lemma 3.3. [1] Let $k + 1 \leq d \leq 2k + 1$. Then the graph $G = C_n^k$ admits b-coloring with d colors.

Theorem 3.4. When 2k+2 divides n, the graph $G = C_n^k \square K_2$ is b-continuous.

Proof. Since the vertices (0, 0), (1, 0), …, (k, 0) are mutually pair wise adjacent in G, $\chi(G) \geq k+1$. By Lemma 3.1, we have $b(G) = 2k + 2$ and hence $S_b(G) \subseteq [k + 1, 2k + 2]$. By Lemma 3.3, C_n^k admits b-coloring with d colors for each d with $k + 1 \leq d \leq 2k + 1$. Hence by Lemma 3.2, G admits b-coloring with d colors for each d with $k + 1 \leq d \leq 2k + 1$ and hence $S_b(G) = [k + 1, 2k + 2]$.

Lemma 3.5. If 2k + 3 divides m, p and $G = C_p^k \square C_m$, then $b(G) = 2k + 3$.

Proof. Consider the graph $H = C_{2k+3}^k \square C_{2k+3}$. First we show that $b(H) = 2k + 3$. Let $V(C_g) = \{1, 2, \ldots, g\}$ be the vertex set of a cycle C_g. Note that, for any vertex $(a, b) \in V(H)$, $|N((a, b))| = 2k+2$ and $N((a,b)) = \{(a \oplus_n 1, b), (a \oplus_n 2, b), \ldots, (a \oplus_n k, b), (a \oplus_n (n-1), b), (a \oplus_n (n-2), b), \ldots, (a \oplus_n (n - k), b), (a, b \oplus_1 1), (a, b \oplus_n (n - 1))\}$, where $\oplus_n$ is the operation, addition modulo $n = 2k + 3$.
Since $\Delta(H) = 2k + 2$, $b(H) = 2k + 3$. It remains to show that H is b-colorable with 2k + 3 colors. Let us color the vertices of H with 2k + 3 colors 1, 2, …, 2k + 3 as follows:
$c((1, 1)) = c((2, 1 \oplus_n 2)) = c((3, 1 \oplus_n 2(2))) = c((4, 1 \oplus_n 2(3))) = \ldots = c((2k + 3, 1 \oplus_n 2(2k + 2))) = 1$,
$c((2, 1)) = c((3, 1 \oplus_n 2)) = c((4, 1 \oplus_n 2(2))) = c((5, 1 \oplus_n 2(3))) = \ldots = c((2k + 3, 1 \oplus_n 2(2k + 1))) = c((1, 1 \oplus_n 2(2k + 2))) = 2$,
$c((3, 1)) = c((4, 1 \oplus_n 2)) = c((5, 1 \oplus_n 2(2))) = c((6, 1 \oplus_n 2(3))) = \ldots = c((2k + 2, 1 \oplus_n 2(2k-1))) = c((2k + 3, 1 \oplus_n 2(2k))) = c((1, 1 \oplus_n 2(2k+1))) = c((2, 1 \oplus_n 2(2k+2))) = 3$,
…,
$c((2k + 3, 1)) = c((1, 1 \oplus_n 2)) = c((2, 1 \oplus_n 2(2))) = c((3, 1 \oplus_n 2(3))) = \ldots = c((2k + 2, 1 \oplus_n 2(2k + 2))) = 2k + 3$.

Note that, in the above coloring, each vertex is colorful and hence $b(H) = 2k + 3$. Consider the graph $G = C_p^k \square C_m$. Define $f : G \to H$ be $f((a, b)) = (a \bmod (2k+ 3), b \bmod (2k+ 3))$. Since 2k +3 divides p and m, f is a covering projection from G onto H. Since H is b-colorable with 2k + 3 colors, by Lemma 2.2, G is also b-colorable with 2k + 3 colors. Since $\Delta(G) = 2k + 2$, $b(G) = 2k + 3$. □

Lemma 3.6. A graph G is b-colorable with d (≥ 3) colors, then so is $G \square K_3$.

Proof. Let G be a b-colorable graph with d colors. Let the corresponding color classes be $C_1, C_2, \ldots, C_d$ with colors 1, 2, …, d respectively. Let $v_1, v_2, \ldots, v_d$ be the colorful vertices of colors 1, 2, …, d respectively. Let $V(K_3) = \{0, 1, 2\}$. Let us color the vertices of $G \square K_3$ as follows. $c((v, 0)) = i$ if $v \in C_i$ for all $1 \leq i \leq d$, $c((v,1))$

= i + 1 if $v \in C_i$ for all $1 \le i \le d-1$ and $c((v, 2)) = i + 2$ if $v \in C_i$ for all $1 \le i \le d-2$. Further, $c((v, 1)) = 1$ if $v \in C_d$, $c((v, 2)) = 1$ if $v \in C_{d-1}$ and $c((v, 2)) = 2$ if $v \in C_d$. From the above coloring, it is easy to observe that the vertices $(v_1, 0), (v_2, 0), \ldots, (v_d, 0)$ are colorful vertices of colors 1, 2, …, d respectively. □

Lemma 3.7. Let n=2(2k+3). Then the graph $G = C_n^k \square K_3$ is b-colorable with 2k+3 colors.

Proof. Let $V(C_n^k) = \{0(= n), 1, 2, \ldots, n-1\}$ be the vertex set of C_n^k and {0, 1, 2} be the vertex set of K_3. It remains to show that G is b-colorable with 2k + 3 colors.
Let $V(G) = V_1 \cup V_2 \cup V_3$, where $V_i = \{(g, i) : g \in V(C_n^k)\}$ for i = 0, 1 and 2.
Let us color the vertices of G with 2k + 2 colors 1, 2, …, 2k + 2 as follows:
Let n = i(2k + 2) + j. We first color the vertices of V_1.
Case 1. If $1 \le j \le k + 1$. Then $c((g, 0)) = g(\mathrm{mod}\ (2k + 2))$ for $1 \le g \le i(2k + 2)$ and $c((g, 0)) = (k + 1) + g$ for $i(2k + 2) \le g \le n$.
Case 2. If $k + 2 \le j \le 2k + 2$. Then $c((g, 0)) = g(\mathrm{mod}\ (2k + 2))$ for $1 \le g \le n$ and $c((g, 0)) = (k + 1) + g$ for $i(2k + 2) \le g \le n$.
Now we color the vertices of V_2 and V_3 as follows:
$c((g,1)) = c((g, 0)) \oplus_{2k+2} (k + 1)$ and $c((g, 2)) = c((g, 0)) \oplus_{2k+2} (k + 2)$. Note that in the above coloring, the vertices (k+1, 0), (k+2, 0), …, (2k+2, 0), ((2k+2)+1, 0), ((2k+2)+2, 0) and ((2k+2)+k, 0) are colorful vertices with colors k + 1, k + 2, …, 2k + 2, 1, 2, …, k respectively. □

Theorem 3.8. When 2k + 3 divides n and $n \ge 2(2k + 3)$, the graph $G = C_n^k \square K_3$ is b-continuous.

Proof. Since the vertices (0, 0), (1, 0), …, (k, 0) are mutually pair wise adjacent, $\chi G) \ge k+1$. As in the proof of Lemma 3.5, we can prove that b(G) = 2k + 3.
Hence, $S_b(G) \subseteq [k+1, 2k+3]$.
By Lemma 3.3 and Lemma 3.6, G is b-colorable with i colors for each i with $k + 1 \le i \le 2k+1$. By Lemma 3.6 and Lemma 3.5, G is b-colorable with i colors for i = 2k+ 2, 2k + 3. Hence $S_b(G) = [k+1, 2k+3]$ and hence G is b-continuous. □

REFERENCES

1. S. Chandra Kumar, T. Nicholas, b-Continuity in Peterson graph and power of a cycle, *International Journal of Modern Engineering Research,*
2. Eric Sopena, Jiaojiao Wu, Coloring the square of the Cartesian product of two cycles, *Discrete Appl. Math.,* 310 (2010), 2327-2333.
3. T. Faik, About the b-continuity of graphs, *Electronic Notes in Discrete Math.,* 17 (2004), 151-156.

4. T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, (2001), Marcel Dekker.
5. R.W. Irving and D.F. Manlove, The b-chromatic number of a graph, *Discrete Appl. Math.,* 91 (1999), 127141.
6. J. Lee, Independent perfect domination sets in Cayley graphs. *J. Graph Theory,* 37(4), (2001), 231-239.
7. C.N. Campos, C.P. De Mello, A result on the total coloring of powers of cycles. *Disc. Appl. Math.,* 155 (2007), 585 – 597.