

Numerical Solutions of Second Order Boundary Value Problems by Galerkin Method with Hermite Polynomials

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Abstract. In this paper, we solve numerically a second order linear boundary value problem, by the technique of Galerkin method. For this, we derive a simple and efficient matrix formulation using Hermite polynomials as trial functions. The proposed method is tested on several numerical examples of second order linear boundary value problems with Neumann and Cauchy types boundary conditions. The approximate solutions of some examples coincide with the exact solutions on using a very few Hermite polynomials. The approximate results, obtained by the propose method, confirm the convergence of numerical solutions and are compared with the existing methods available in the literature.

Keywords: Second order boundary value problems, Galerkin method, Hermite polynomials.

AMS Mathematics Subject Classification (2010): 42A10, 42A15

1. Introduction

There are many linear and nonlinear problems in science and engineering, namely second order differential equations with various types of boundary conditions, are solved either analytically or numerically. Numerical simulation in engineering science and in applied mathematics has become a powerful tool to model the physical phenomena, particularly when analytical solutions are not available then very difficult to obtain. In the literature of numerical analysis solving second order boundary value problem (BVP) of differential equations, many authors have attempted to obtain higher accuracy rapidly by using a numerous methods. Such that diffusion occurring in the presence of exothermic chemical reaction, heat conductions associated with radiation effect [1]. Solving such type of boundary

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value problems analytically is possible only in very rare cases. Study in this field is very interesting. Various methods are available in the literature concerning their numerical solutions [2-4]. Khan in [5] obtained a parametric cubic spline solution of two point boundary value problems, in [6] Feng and Li solved a second-order Neumann boundary value problem with singular nonlinearity for exact three positive solutions, in [7] Lima and Carpentier obtained a Numerical solution of a singular boundary-value problem in non-Newtonian fluid mechanics, in [8] Rashidinia and Jalilian introduced a spline solution of two point boundary value problems, in [9] Viswanadham et al. obtained a numerical solution of a fourth order boundary value problems by Galerkin method with Quintic B-splines basis, in [10] Das et al. produced a method for solutions of nonlinear second order multi-point boundary value problems and recently in [11] Bhatti and Bracken solved linear and non-linear differential equation numerically by Galerkin method in a Bernstein polynomials basis.

However, in this paper a very simple and efficient Galerkin numerical method is proposed with Hermite polynomials as trial functions. The formulation is derived to solve second order boundary value problem with two different cases of boundary conditions, in details, in Section 3. In Section 2, we give a short introduction of Hermite polynomials. Finally, two examples of Neumann boundary value problems and one example of Cauchy boundary value problems are given to verify the proposed formulation. The results of each example indicate the convergence numerical solutions. Moreover, this method can provide even the exact solutions, with a few lower order Hermite polynomials, if the equation is simple.

2. Hermite Polynomials

The general form of the Hermite polynomials of n th degree is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) ,$$
$$n = 0, 1, 2, 3, \dots \dots \dots (1)$$

Using **MATLAB** code, the first few Hermite polynomials are given below:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = -2 + 4x^2$$

$$H_3(x) = -(12x - 8x^3)$$

$$H_4(x) = (12 - 48x^2 + 16x^4)$$

$$H_5(x) = (120x - 160x^3 + 32x^5)$$

$$H_6(x) = -(120 - 720x^2 + 480x^4 - 64x^6)$$

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Now the first six Hermite polynomials over the interval $[-1, 1]$ are shown in Figure 1.

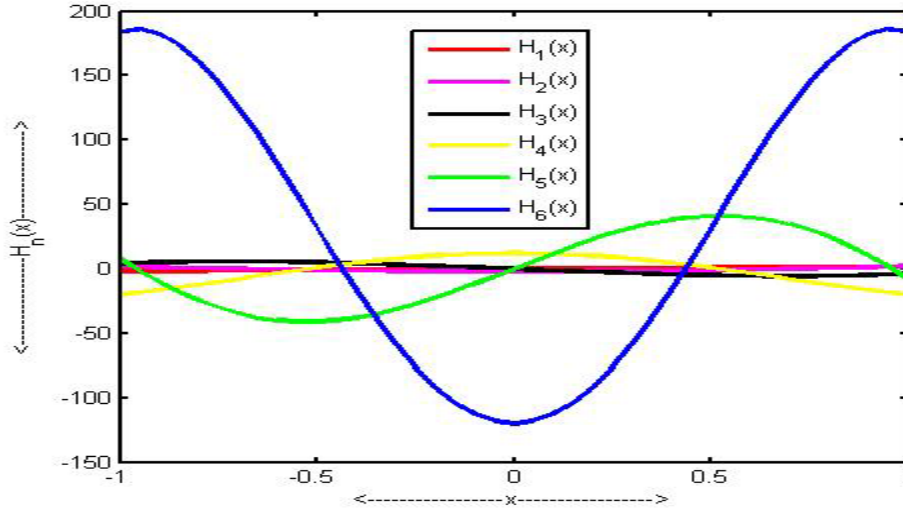


Figure 1. Graph of first 6 Hermite polynomials over the interval $[-1, 1]$

3. Formulation of Boundary Value Problems in Matrix Form

We consider a general second order linear boundary value problem are given by

$$a_2(x) \frac{d^2 u}{dx^2} + a_1(x) \frac{du}{dx} + a_0(x)u = g(x) ;$$

$$a \leq x \leq b \tag{2}$$

subject to the boundary conditions

$$u(a) = \gamma_1, \quad u'(b) = \gamma_2$$

where $a_2(x)$, $a_1(x)$, $a_0(x)$ and $g(x)$ are given functions and γ_1, γ_2 are given constants and $u(x)$ is the unknown function or exact solution of (2), which is to be determined.

To use Hermite polynomials over $[0, 1]$, we convert each BVP to an equivalent BVP on $[0, 1]$. The BVP can be converted to an equivalent problem on $[0, 1]$ by letting $x = (b - a)x + a$. Then the Equation (2) is equivalent to the BVP

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$$\widetilde{a}_2(x) \frac{d^2 u}{dx^2} + \widetilde{a}_1(x) \frac{du}{dx} + \widetilde{a}_0(x) u = \widetilde{g}(x),$$

$$0 \leq x \leq 1 \tag{3}$$

Subject to the boundary conditions

$$u(0) = \gamma_1, \quad u'(1) = \gamma_2 \tag{3a}$$

where

$$\widetilde{a}_2(x) = \frac{1}{(b-a)^2} a_2((b-a)x + a), \quad \widetilde{a}_1(x) = \frac{1}{(b-a)} a_1((b-a)x + a),$$

$$\widetilde{a}_0(x) = a_0((b-a)x + a) \text{ and } \widetilde{g}(x) = g((b-a)x + a).$$

Now we use the technique of Galerkin method [Lewis, 2] to find an approximate solution $\tilde{u}(x)$ of (3). For this, we assume that

$$\tilde{u}(x) = \sum_{i=0}^n c_i N_i(x) \tag{4}$$

where $N_i(x)$ are piecewise polynomials, namely –Hermite polynomials of degree i and c_i are unknown parameters, to be determined. Applying Galerkin method with the basis functions $N_j(x)$, we get

$$\int_0^1 \left[\widetilde{a}_2(x) \frac{d^2 \tilde{u}}{dx^2} + \widetilde{a}_1(x) \frac{d\tilde{u}}{dx} + \widetilde{a}_0(x) \tilde{u} \right] N_j(x) dx$$

$$= \int_0^1 \widetilde{g}(x) N_j(x) dx \tag{5}$$

Integrating first term by parts on the left hand side of (5), we get after applying the Neumann conditions prescribed in (3a) as

$$\begin{aligned}
 \int_0^1 \left[-\frac{d\tilde{u}}{dx} \frac{d}{dx} [\widetilde{a_2}(x)N_j(x)] + \widetilde{a_1}(x) \frac{d\tilde{u}}{dx} N_j(x) + \widetilde{a_0}(x)\tilde{u}N_j(x) \right] dx \\
 = \int_0^1 \tilde{g}(x)N_j(x) dx + \widetilde{a_2}(0)\gamma_1 N_j(0) \\
 - \widetilde{a_2}(1)\gamma_2 N_j(1)
 \end{aligned} \tag{6}$$

By substituting (4) into (6), we get a system of equation in matrix form as

$$\sum_{i=0}^n c_i K_{i,j} = F_j ; j = 0, 1, 2, \dots, n \tag{7}$$

where

$$\begin{aligned}
 K_{i,j} = \int_0^1 \left[-N_i'(x) \frac{d}{dx} [\widetilde{a_2}(x)N_j(x)] + \widetilde{a_1}(x)N_i'(x)N_j(x) \right. \\
 \left. + \widetilde{a_0}(x)N_i(x)N_j(x) \right] dx \\
 F_j = \int_0^1 \tilde{g}(x)N_j(x) dx + \widetilde{a_2}(0)\gamma_1 N_j(0) \\
 - \widetilde{a_2}(1)\gamma_2 N_j(1)
 \end{aligned}$$

Now the unknown parameters c_i are determined by solving the system of equations (7) and substituting these values of parameters in (4), we get the approximate solution $\tilde{u}(x)$ of the equation (3) for Neumann boundary conditions.

Again, Integrating first term by parts on the left hand side of (5), we get after applying the Cauchy conditions prescribed in (3a) as

$$\begin{aligned}
 \int_0^1 \left[-\frac{d\tilde{u}}{dx} \frac{d}{dx} [\widetilde{a_2}(x)N_j(x)] + \widetilde{a_1}(x) \frac{d\tilde{u}}{dx} N_j(x) + \widetilde{a_0}(x)\tilde{u}N_j(x) \right] dx \\
 = \int_0^1 \tilde{g}(x)N_j(x) dx \\
 - \widetilde{a_2}(1)\gamma_2 N_j(1)
 \end{aligned} \tag{8}$$

By substituting (4) into (8), we get a system of equation in matrix form as

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$$\sum_{i=0}^n c_i K_{i,j} = F_j ; \quad j = 0, 1, 2, \dots, n \quad (9)$$

where

$$K_{i,j} = \int_0^1 \left[-N_i'(x) \frac{d}{dx} [\widetilde{a}_2(x) N_j(x)] + \widetilde{a}_1(x) N_i'(x) N_j(x) \right. \\ \left. + \widetilde{a}_0(x) N_i(x) N_j(x) \right] dx$$

$$F_j = \int_0^1 \widetilde{g}(x) N_j(x) dx$$

$$- \widetilde{a}_2(1) \gamma_2 N_j(1)$$

Now the unknown parameters c_i are determined by solving the system of equations (9) and substituting these values of parameters in (4), we get the approximate solution $\tilde{u}(x)$ of the equation (3) for Cauchy boundary conditions. Then the absolute error between exact and approximate solutions is obtained by using the following formula $|u(x) - \tilde{u}(x)|$

4. Numerical Examples

In this section, we explain three examples of Neumann boundary value problems and one example of Cauchy boundary value problems which are available in the literature. For each example we find the approximate solutions using same number of piecewise Hermite polynomials. The computations, associated with the examples, are performed by **MATLAB** [13, 14].

Example 1. Consider the linear boundary value problem

$$\frac{d^2 u}{dx^2} + u = x^2 e^{-x} ; \quad 0 \leq x \leq 10 \quad (10)$$

Subject to the Neumann boundary conditions

$$u'(0) = 0, \quad u'(10) = 0$$

The exact solution is

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$$u(x) = -\frac{(\cos 10 + 99e^{-10})}{2 \sin 10} \cos x - \frac{1}{2} \sin x + \frac{1}{2} e^{-x} (1+x)^2$$

Result has been shown for different values of x in Table 1 for $n = 12$. Also Figure 2 shows the exact and approximate solution. The maximum absolute errors obtained by the proposed method in the order of 10^{-1} . On the contrary, the accuracy is found nearly the order of 10^{-5} for $n = 20$ in .

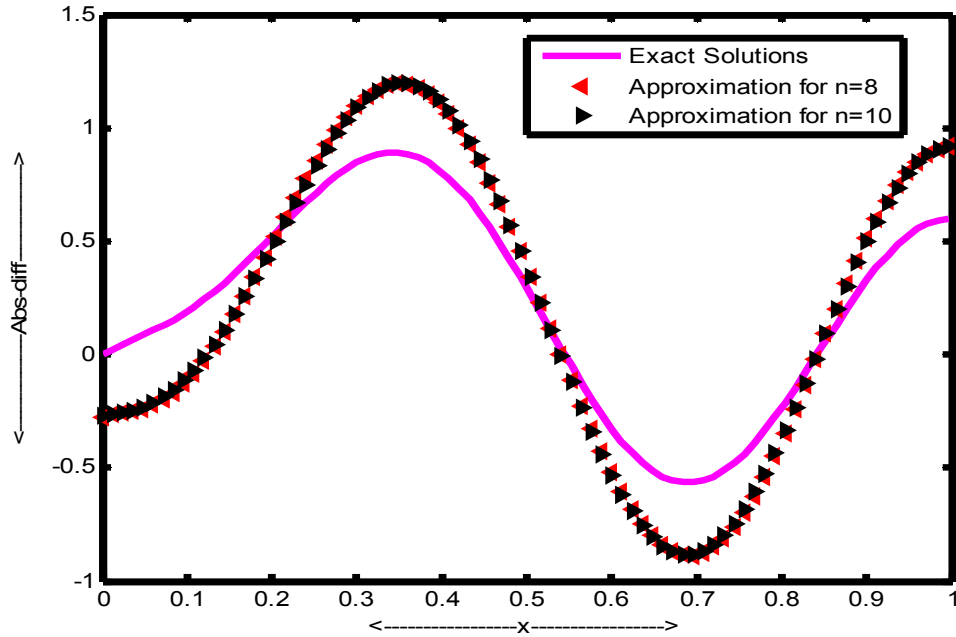


Figure 2. Exact solutions and Numerical solutions for Example 1

Example 2. Consider the linear boundary value problem

$$\frac{d^2 u}{dx^2} + u = \cos x; \quad 0 \leq x \leq 5 \quad (11)$$

Subject to the Neumann boundary conditions

$$u'(0) = 0, \quad u'(5) = 0$$

The exact solution is

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$$u(x) = \frac{1}{4}(-6\cos^2 5 + 2 + 10 \cot 5 + \cot 5 \sin 10 + 2\cos 10) \cos x \\ + \frac{1}{4}(2\cos^3 x + 2x \sin x + \sin x \sin 2x)$$

Result has been shown for different values of x in Table 1 for $n = 12$. Also Figure 3 shows the exact and approximate solution. The maximum absolute errors obtained by the proposed method in the order of 10^{-1} .

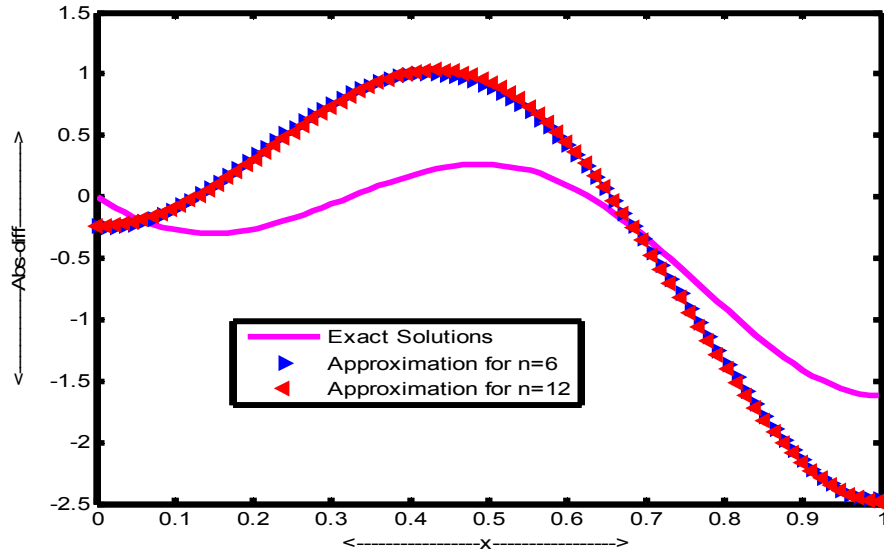


Figure 3. Exact solutions and Numerical solutions for Example 2

Table 1: Computed Absolute Error of examples 1 and 2.

	Example 1 for n=12			Example 2 for n=12		
x	Exact Solutions	Approx. Solutions	Absolute Error	Exact Solutions	Approx. Solutions	Absolute Error
0.1	0.1905656	-0.0993485	1.5213348E+00	-0.2683548	-0.0878111	6.7277990E-01
0.2	0.5198704	0.4734712	8.9251515E-02	-0.2606393	0.2949467	2.1316277E+00
0.3	0.8471665	1.0871529	2.8328132E-01	-0.0595928	0.7350613	1.3334725E+01
0.4	0.8031348	1.1087492	3.8052691E-01	0.1730007	1.0120826	4.8501662E+00
0.5	0.2928847	0.3830705	3.0792232E-01	0.2634819	0.9412174	2.5722287E+00

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0.6	-0.3280263	-0.5359707	6.3392590E-001	0.0974091	0.4471576	3.5905097E+00
0.7	-0.5625127	-0.8776264	5.6018947E-001	-0.3298266	-0.3948662	1.9719354E-01
0.8	-0.2370441	-0.3694783	5.5869004E-001	-0.9007898	-1.3663642	5.1685119E-01
0.9	0.3270311	0.4989862	5.2580667E-001	-1.4078951	-2.1621571	5.3573739E-01
1.0	0.6001005	0.9183684	5.3035772E-001	-1.6208290	-2.4816165	5.3107857E-01

Example 3. Consider the linear boundary value problem

$$\frac{d^2u}{dx^2} + u = -x; \quad 0 \leq x \leq 2 \quad (12)$$

Subject to the Cauchy boundary conditions

$$u(0) = 0, \quad u'(2) = 0$$

The exact solution is

$$u(x) = \frac{\sin x}{\cos 2} - x$$

Result has been shown for different values of x in Table 2 for $n = 12$. Also Figure 4 shows the exact and Absolute Error. The maximum absolute errors obtained by the proposed method in the order 10^{-16} in Hermite basis.

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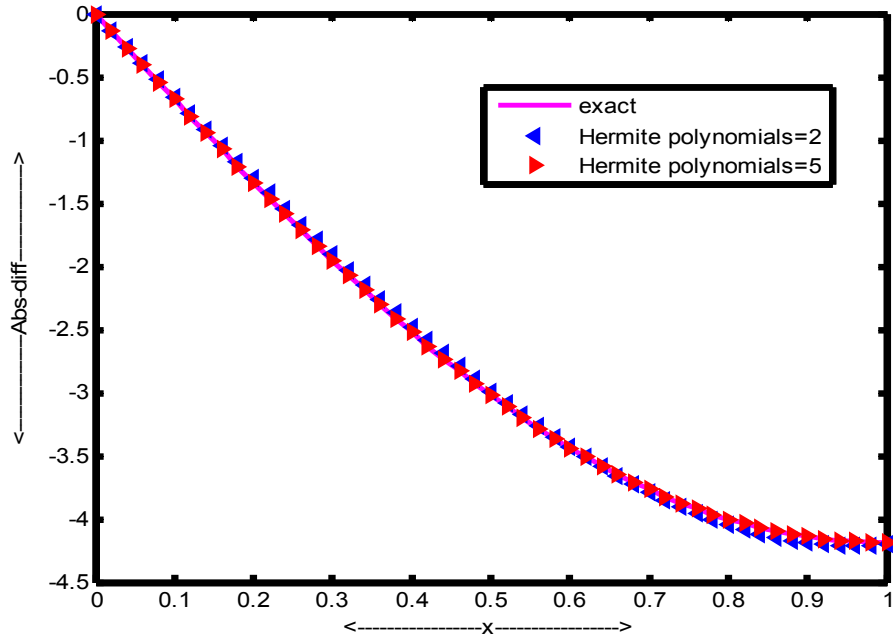


Figure 4. Exact solutions and Numerical solutions for Example 3

Table 1: Computed Absolute Error of Example 3.

Example 3 for n=12					
x	Exact Solutions	Absolute Error	x	Exact Solutions	Absolute Error
0.1	-0.6774020	0.0000000E+000	0.6	-3.4396880	0.0000000E+000
0.2	-1.3357715	0.0000000E+000	0.7	-3.7680337	1.1785702E-016
0.3	-1.9568347	2.2694263E-016	0.8	-4.0019733	0.0000000E+000
0.4	-2.5238052	1.7596018E-016	0.9	-4.1401539	0.0000000E+000
0.5	-3.0220531	1.4694951E-016	1.0	-4.1850399	0.0000000E+000

5. Conclusions

In this paper, we have developed Galerkin method to approximate the solution of second order Neumann and Cauchy boundary value problems. It is observed that the approximate results converge monotonically to the exact solutions. We also notice

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that the approximate solutions coincide with the exact solutions even a few of the polynomials are used in the approximation which are shown in Table 1 and Table 2. We may realize that this method may be applied to solve other linear differential equations for the desired accuracy. The objective of this paper is to present a simple and accurate method to solve second order boundary value problems.

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