Annals of Pure and Applied Mathematics
Vol. 1, No. 2, 2012, 158-167
ISSN: 2279-087X (P), 2279-0888(online)
Published on 16 November 2012
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Numerical Solutions of Volterra Integral Equations of Second kind with the help of Chebyshev Polynomials

M. M. Rahman[1*], M. A. Hakim[4], M. Kamrul Hasan[2], M. K. Alam[2] and L. Nowsher Ali[3]

[1] *Department of Mathematics, Islamic University, Kushita-7003, Bangladesh.*
Email: mizan_iu@yahoo.com
[2] *Department of Mathematics & Statistics, Bangladesh University of Business & Technology, Dhaka – 1216, Bangladesh.*
[3] *Department of* Electrical and Electronic Engineering*, Green University of Bangladesh, Dhaka-1216, Bangladesh.*
[4] *Department of Mathematics, Comilla University, Comilla, Bangladesh.*

Received 23 September2012; accepted 30 September 2012

Abstract. In the present paper, we solve numerically Volterra integral equations of second kind with regular and singular kernels, by the well known Galerkin weighted residual method. For this, we derive a simple and efficient matrix formulation using Chebyshev polynomials as trial functions. Numerical examples are considered to verify the effectiveness of the proposed derivations and numerical solutions are compared with the existing methods available in the literature.

***Keywords*:** Volterra integral equations, Galerkin method, Chebyshev polynomials.

AMS Mathematics Subject Clasifications (2010): 42A10, 42A15

1. Introduction

Many problems of mathematical physics can be started in the form of integral equations. These equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. Therefore, the study of integral equations and methods for solving them are very useful in application. In recent years, there has been a growing interest in the Volterra integral equations arising in various fields of physics and engineering [1], e.g., potential theory and Dirichlet problems, electrostatics, the particle transport problems of astrophysics and reactor theory, contact problems, diffusion problems, and heat transfer problems.

Some valid numerical methods, for solving Volterra equations using various polynomials [2], have been developed by many researchers. Very recently, Maleknejad *et al* [3] and Mandal and Bhattacharya [4] used Bernstein polynomials in approximation techniques, Shahsavaran [5] solved by Block – Pulse functions and Taylor Expansion method. Taylor polynomials were also used by Bellour and Rawashdeh [6] and Wang [7] with computer algebra. Bernstein polynomials were used for the solution of second order linear and first order non-linear differential equations by Bhatti and Bracken [8]. These polynomials have been also used for solving Fredholm integral equations of second kind by Shirin and Islam [9]. Babolian and Delves [10] describe an augmented Galerkin technique for the numerical solution of first kind Fredholm integral equations. In [11] a numerical solution of Fredholm integral equations of the first kind via piecewise interpolation is proposed. Lewis [12] studied a computational method to solve first kind integral equations.

However, in this paper a very simple and efficient Galerkin weighted residual numerical method is proposed with Chebyshev polynomials as trial functions. The formulation is derived to solve the linear Volterra integral equations of second kind having regular as well as weakly singular kernels, in details, in Section 3. In Section 2, we give a short introduction of Chebyshev polynomials. Finally, five examples of different kinds of Volterra integral equations are given to verify the proposed formulation. The results of each example indicate the convergence numerical solutions. Moreover, this method can provide even the exact solutions, with a few lower order Chebyshev polynomials, if the equation is simple.

2. Chebyshev Polynomials

The general form of the Chebyshev polynomials [12] of *n*th degree is defined by

$$T_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]}(-1)^m \frac{n!}{(2m)!(n-2m)!}(1-x^2)^m x^{n-2m} \tag{1}$$

Using ***MATLAB*** code, the first few Chebyshev polynomials from equation (1) are given below:

$T_0(x) = 1$, $T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_5(x) = 16x^5 - 20x^3 + 5x$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

Now the first six Chebyshev polynomials over the interval $[-1, 1]$ are shown in Fig. 1.

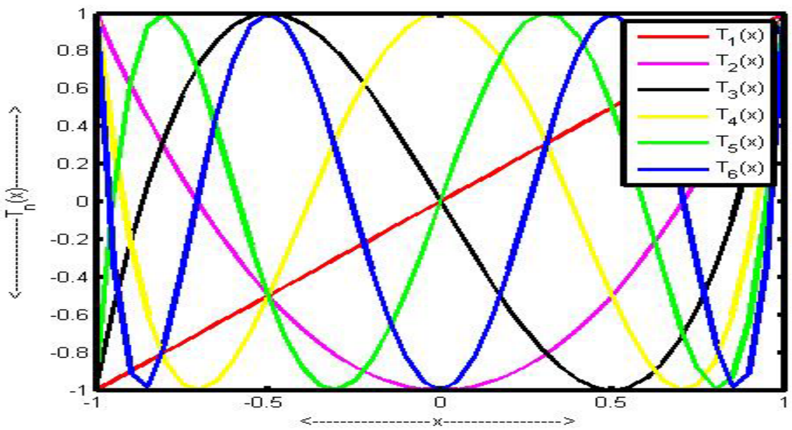

Figure 1. Graph of first 6 Chebyshev polynomials over the interval [-1, 1]

3. Formulation of Integral Equation in Matrix Form

We consider the Volterra integral equation (VIE) of the first kind [6] given by

$$\int_a^x k(x,t)\varphi(t)dt = f(x), \qquad a \le x \le b \tag{2}$$

where $\varphi(x)$ is the unknown function, to be determined, $k(x,t)$, the kernel, is a continuous or discontinuous and square integrable function, $f(x)$ being the known function satisfying $f(a) = 0$.

Now we use the technique of Galerkin method, [Lewis, 11], to find an approximate solution $\tilde{\varphi}(x)$ of (2). For this, we assume that

$$\tilde{\varphi}(x) = \sum_{i=0}^{n} a_i T_i(x) \tag{3}$$

where $T_i(x)$ are Chebyshev polynomials of degree i defined in equation (1) and a_i are unknown parameters, to be determined. Substituting (3) into (2), we get

$$\sum_{i=0}^{n} a_i \int_a^x k(x,t)T_i(t)dt = f(x), \qquad a \le x \le b \tag{4}$$

Then the Galerkin equations are obtained by multiplying both sides of (4) by $T_j(x)$ and then integrating with respect to x from $a\ to\ b$, we have

$$\sum_{i=0}^{n} a_i \int_a^b \left[\int_a^x K(x,t)T_i(t)dt\right] T_j(x)dx$$

$$= \int_a^b f(x)\, T_j(x), \tag{5}$$

j=0, 1, 2, … n.

Since in each equation, there are two integrals. The inner integrand of the left side is a function of x, and t, and is integrated with respect to t from $a\ to\ x$. As a result the outer integrand becomes a function of x only and integration with respect to x from $a\ to\ b$ yields a constant. Thus for each $j = 0,1,2,\cdots\cdots,n$ we have a linear equation with $n+1$ unknowns a_i, $i = 0,1,2,\cdots\cdots,n$.

Finally (5) represents the system of $n+1$ linear equations in $n+1$ unknowns, are given by

$$\sum_{i=0}^{n} a_i K_{i,j} = F_j\ ; i,j = 0,1,2,\dots n$$

where

$$K_{i,j} = \int_a^b \left[\int_a^x K(x,t)T_i(t)dt\right] T_j(x)dx,$$

$$i,j = 0,1,2,\cdots\cdots,n$$

$$F_j = \int_a^b f(x)T_j(x)\, dx,$$

$$j = 0,1,2,\cdots\cdots,n$$

Now the unknown parameters a_i are determined by solving the system of equations (6) and substituting these values of parameters in (3), we get the approximate solution $\tilde{\varphi}(x)$ of the integral equation (2).

Now, we consider the Volterra integral equation (VIE) of the second kind [6] given by

$$c(x)\varphi(x) + \lambda \int_a^x k(x,t)\varphi(t)dt = f(x),$$

$$a \le x \le b \tag{7}$$

where $\varphi(x)$ is the unknown function to be determined, $k(x,t)$, the kernel, is a continuous or discontinuous and square integrable function, $f(x)$ and $c(x)$ being the known function and λ is the constant. Then applying the same procedure as described above, we obtain

$$\sum_{i=0}^{n} a_i K_{i,j} = F_j\ ;\ i,j = 0,1,2,\dots,n$$

$$K_{i,j} = \left[\int_a^b \left[c(x)T_i(x) + \lambda \int_a^x K(x,t)T_i(t)dt \right] T_j(x)dx \right], i,j = 0,1,\dots,n$$

where

$$F_j = \int_a^b f(x)T_j(x)\,dx, j = 0,1,2,\dots,n \tag{8}$$

Now the unknown parameters a_i are determined by solving the system of equations (8) and substituting these values of parameters in (3), we get the approximate solution $\tilde{\varphi}(x)$ of the integral equation (7). The maximum absolute error for this formulation is defined by

Maximum absolute error $= Max\ |\varphi(x) - \tilde{\varphi}(x)|$

4. Numerical Examples

In this paper, we illustrate the above mentioned methods with the help of five numerical examples, which include second kind with regular kernels and weakly singular kernels, available in the existing literature [1- 5]. The computations, associated with the examples, are performed by **MATLAB** [13, 14]. The convergence of each linear Volterra integral equations is calculated by

$E = |\,\tilde{\varphi}_{n+1}(x) - \tilde{\varphi}_n(x)| \prec \delta$

where $\tilde{\varphi}_n(x)$ denotes the approximate solution by the proposed method using *n*th degree polynomial approximation and δ varies from 10^{-6} for $n \geq 10$.

Example 1. Consider the Volterra integral equations of second kind

$$\varphi(x) = x + \int_0^x (t-x)\varphi(t)dt \qquad 0 \leq x \leq 1 \tag{9}$$

The exact solution is $\varphi(x) = \sin x$. Results have been shown in Table 1 for $n = 1,\ 3,\ 10$. Also Fig. 1 shows the exact and approximate solution for $n = 1,\ 3\ \ and\ 10$. The maximum absolute errors obtain in the order of 10^{-13} for $n = 10$.

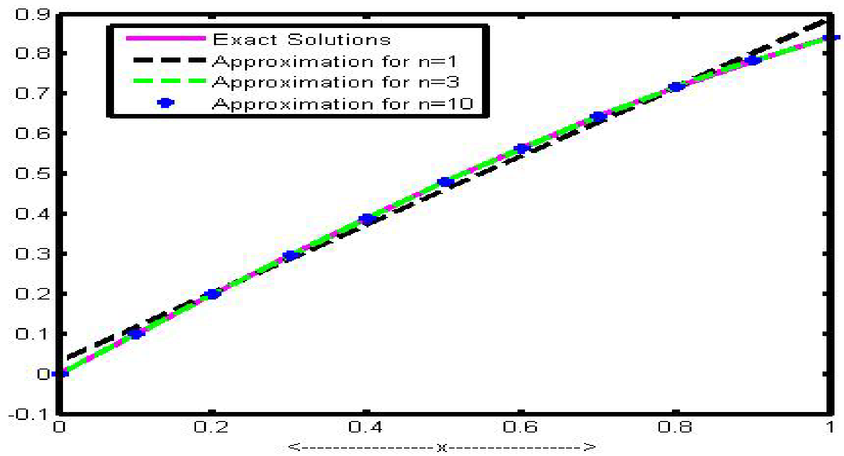

Figure 2. Exact solution and Numerical solution of example 1 for $n = 1,\ 3,\ 10$.

Example 2. Consider the Volterra integral equations of second kind

$$\varphi(x) = e^{x} + \int_{0}^{x} \varphi(t)dt \qquad 0 \le x \le 1 \tag{10}$$

The exact solution is $\varphi(x) = e^{x}(1+x)$. Results have been shown in Table 1 for $n = 1,\ 2,\ 3$. Also Fig. 2 shows the exact and approximate solution for $n = 1,\ 2\ and\ 3$. The maximum absolute errors obtain in the order of 10^{-4} for $n = 2$

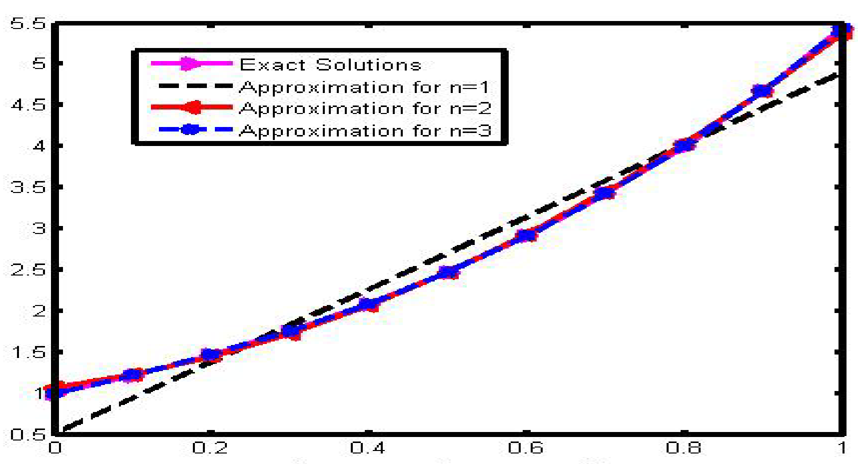

Figure 3. Exact solution and Numerical solution of example 2 for $n = 1,\ 2,\ 3$

Table 1. Computed Absolute Error of examples 1 and 2.

	Example 1 for n=10			**Example 2 for n=3**		
x	**Exact Solutions**	**Approx. Solutions**	**Absolute Error**	**Exact Solutions**	**Approx. Solutions**	**Absolute Error**
0.0	0.0000000	0.0000000	Inf-0.0000000	1.0000000	0.9945330	5.4670224E-003
0.1	0.0998334	0.0998334	2.0508063E-012	1.2156880	1.2167930	9.0893339E-004
0.2	0.1986693	0.1986693	8.3558996E-013	1.4656833	1.4679684	1.5590778E-003
0.3	0.2955202	0.2955202	2.0756558E-013	1.7548164	1.7556648	4.8342409E-004
0.4	0.3894183	0.3894183	2.4960310E-013	2.0885546	2.0874875	5.1090843E-004
0.5	0.4794255	0.4794255	3.6565473E-013	2.4730819	2.4710422	8.2478268E-004
0.6	0.5646425	0.5646425	1.5317015E-013	2.9153901	2.9139342	4.9939072E-004
0.7	0.6442177	0.6442177	1.1908461E-013	3.4233796	3.4237690	1.1375748E-004
0.8	0.7173561	0.7173561	2.5211375E-013	4.0059737	4.0081523	5.4383673E-004
0.9	0.7833269	0.7833269	2.8431391E-013	4.6732459	4.6746893	3.0887023E-004
1.0	0.8414710	0.8414710	8.8095244E-013	5.4365637	5.4309857	1.0260007E-003

Example 3. Consider the Volterra integral equations of second kind

$$\varphi(x) + \int 3^{x-t}\varphi(t)dt = x3^{x} \qquad 0 \le x \le 1 \qquad (11)$$

The exact solution is $\varphi(x) = 3^{x}(1-e^{-x})$. Using the formula derived in the previous section and solving the system (8) for $n \geq 2$, we get the approximate solution is $\widetilde{\varphi}(x) = 3^{x}(1-e^{-x})$, which is the exact solution. On the contrary, the accuracy is found nearly the order of 10^{-3} for $n = 3$.

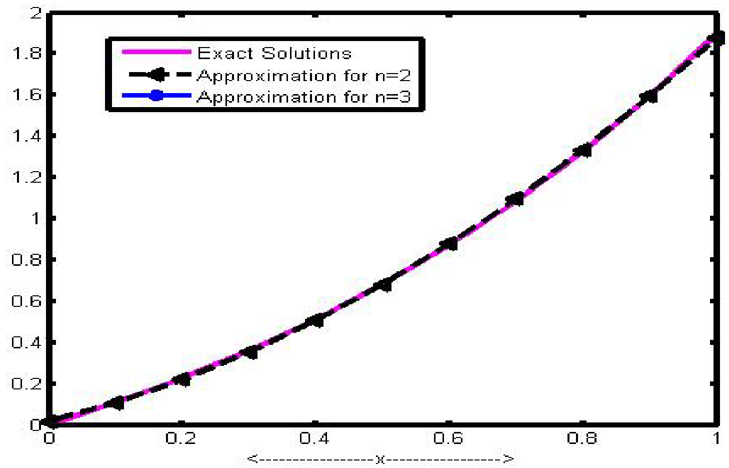

Figure 4. Exact solution and Numerical solution of example 3 for $n = 2, 3$

Example 4. Consider the weakly singular Volterra integral equations of second kind

$$\varphi(x) - \int_0^x \frac{1}{(x-t)^{1/2}} \varphi(t)dt = x^7 (1 - \frac{4096}{6435} x^{1/2}) \quad 0 \le x \le 1 \tag{12}$$

The exact solution is $\varphi(x) = x^7$. Using the formula derived in the previous section and solving the system (8) for $n \ge 4$, we get the approximate solution is $\widetilde{\varphi}(x) = x^7$, which is the exact solution. On the contrary, the accuracy is found nearly the order of 10^{-2} for $n = 4$ in [4].

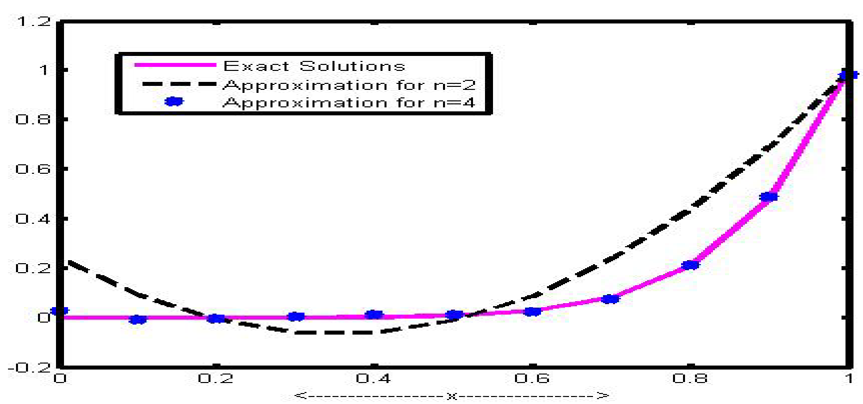

Figure 5. Exact solution and Numerical solution of example 4 for $n = 2, \ 4$

Table 2. Computed Absolute Error of examples 3 and 4.

	Example 3 for n=2			Example 4 for n=3		
x	**Exact Solutions**	**Approx. Solutions**	**Absolute Error**	**Exact Solutions**	**Approx. Solutions**	**Absolute Error**
0.0	0.0000000	0.0166303	Inf-0.0074113	0.0000000	0.0262542	Inf-0.0074113
0.1	0.1062132	0.1074453	1.1600483E-002	0.0000001	-0.0088862	8.8863240E+004
0.2	0.2258127	0.2193524	2.8608994E-002	0.0000128	-0.0067155	5.2565073E+002
0.3	0.3603635	0.3523517	2.2232608E-002	0.0002187	0.0042983	1.8653831E+001
0.4	0.5116124	0.5064431	1.0103823E-002	0.0016384	0.0101804	5.2136198E+000
0.5	0.6815089	0.6816267	1.7285379E-004	0.0078125	0.0114491	4.6548286E-001
0.6	0.8722292	0.8779024	6.5041788E-003	0.0279936	0.0231158	1.7424592E-001
0.7	1.0862024	1.0952702	8.3481474E-003	0.0823543	0.0746853	9.3122623E-002
0.8	1.3261396	1.3337302	5.7238171E-003	0.2097152	0.2101551	2.0977552E-003
0.9	1.5950668	1.5932823	1.1187893E-003	0.4782969	0.4880164	2.0321071E-002
1.0	1.8963617	1.8739265	1.1830649E-002	1.0000000	0.9812532	1.8746832E-002

Example 5. Consider the weakly singular Volterra integral equations of second kind

$$\varphi(x)+\int_0^x \frac{1}{(x-t)^{1/2}}\varphi(t)dt = x^2+\frac{16}{15}x^{1/2} \qquad 0 \le x \le 1 \tag{13}$$

The exact solution is $\varphi(x)=x^2$. Using the formula derived in the previous section and solving the system (8) for $n \ge 2$, we get the approximate solution is $\widetilde{\varphi}(x)=x^2$, which is the exact solution.

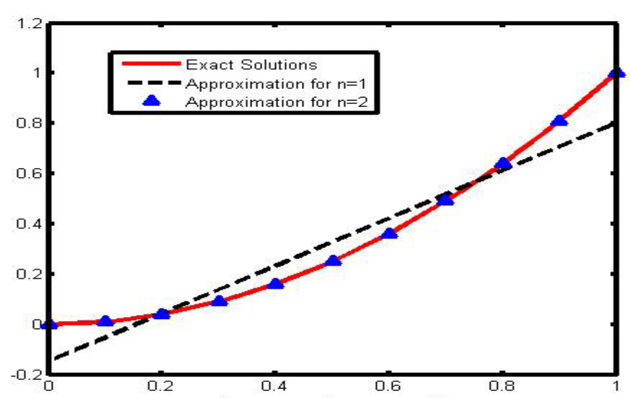

Figure 6. Exact solution and Numerical solution of example 5 for $n = 1$ *and* 2

5. Conclusions

In this paper, a very simple and efficient Galerkin weighted residual method based on the Chebyshev polynomial basis has been developed to solve second kind and also singular Chebyshev integral equations. The numerical results obtained by the proposed method are in good agreement with the exact solutions. In this paper, we may note that the numerical solutions coincide with the exact solutions even a few of the polynomials are used in the approximation, which are shown in example 1-5. We also notice that the accuracy increase with increase the number of polynomials in the approximations, which is shown in Table 1 and Table 2. We may realize that this method may be applied to solve other integral equations for the desired accuracy.

REFERENCES

1. Abdul J. Jerri, *Introduction to Integral Equations with Applications,* John Wiley & Sons Inc., (1999).
2. N. Saran, S. D. Sharma and T. N. Trivedi, *Special Functions,* Seventh edition, Pragati Prakashan, (2000).
3. K. Maleknejad, E. Hashemizadeh and R. Ezzati, A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation, *Commun. Nonlinear Sci. Numer. Simulat,* 16 (2011), 647-655.
4. B. N. Mandal and Subhra Bhattacharya, Numerical solutions of some classes of integral equations using Bernstein polynomials, *Appl. Mathe. Comput,* 190 (2007), 1707-1716.
5. A. Shahsavaran, Numerical approach to solve second kind Volterra integral equations of Abel type using Block-Pulse functions and Taylor expansion by collocation method, *Appl. Mathe. Sci.*, 5 (2011), 685 – 696.
6. Azzeddine Bellour and E. A. Rawashdeh, Numerical solution of first kind integral equations by using Taylor polynomials, *J. Inequal. Speci. Func,* 1 (2010), 23-29.
7. Weiming Wang, A mechanical algorithm for solving the Volterra integral equation, *Appl. Mathe. Comput.,* 172 (2006), 1323 – 1341.
8. M. Idrees Bhatti and P. Bracken, Solutions of differential equations in a Bernstein polynomial basis, *J. Comput. Appl. Mathe*., 205 (2007), 272 – 280.
9. A. Shirin and M. S. Islam, Numerical Solutions of Fredholm integral equations using Bernstein polynomials, *J. Sci. Res.*, 2 (2) (2010), 264 – 272.
10. E. Babolian and L. M. Delves, An augmented Galerkin method for first kind Fredholm equations, *Journal of the Institute of Mathematics and Its Applications*, 24(2) (1979), 157–174.
11. G. Hanna, J. Roumeliotis, and A. Kucera, Collocation and Fredholm integral equations of the first kind, *Journal of Inequalities in Pure and Applied Mathematics*, 6(5) (2005), 1–8.
12. B. A. Lewis, On the numerical solution of Fredholm integral equations of the first kind, *Journal of the Institute of Mathematics and Its Applications*, 16(2) (1975), 207–220.
13. Steven C. Chapra, *Applied Numerical Methods with MATLAB for Engineers and Scientists,* Second edition, Tata McGraw-Hill, (2007).
14. Stephen J. Chapman, *MATLAB Programming for Engineers*, Third edition, Thomson Learning, (2004).