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Some Properties of 0-Distributive and 1-Distributive Lattices

Razia Sultana[1], Md. Ayub Ali[2] and A. S. A. Noor[3]

[1]Department of Mathematics
Govt. Titumir College, Dhaka, Bangladesh.
Email: raziabd.sultana@gmail.com
[2]Department of Mathematics
Jagannath University, Dhaka, Bangladesh.
Email: ayub_ju@yahoo.com
[3]Department of ECE,
East West University, Dhaka, Bangladesh.
E mail: noor@ewubd.edu

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Abstract. In this paper we have included several properties of 0-distributive and 1-distributive lattices. We have provided many characterizations of 1-distributive lattices. We also studied the 1-distributive lattices by using a prime Separation /;[theorem. We prove that a lattice L which is both 0 and 1- distributive, is complemented if and only if its prime ideals are unordered. We also show that a 0-distributive complemented 0- distributive lattice is 0-modular if and only if it is weakly complemented. Finally we include some results on semi prime filters.

***Keywords**:* 0-distributive lattice,1-distributive lattice, Annihilator ideal, Dual annihilator ideal.

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1. Introduction

In generalizing the notion of pseudo complemented lattice, J. C. Varlet [5] introduced the notion of 0-distributive lattices. Then [1] have given several characterizations of these lattices. On the other hand, [3] have studied them in meet semi lattices. A lattice L with 0 is called a *0-distributive* lattice if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. Of course every distributive lattice is 0-distributive. 0-distributive lattice L can be characterized by the fact that the set of all elements disjoint to $a \in L$ forms an ideal. So every pseudo complemented lattice is 0-distributive. Similarly, a lattice L with 1 is called a 1-distributive lattice if for all $a,b,c \in L$ with $a \vee b = 1 = a \vee c$ imply $a \vee (b \wedge c) = 1$. Thus it can be characterized by the fact that the set of all elements whose join with the element a is equal to 1, forms a filter. Hence every dual pseudo complemented lattice is 1-distributive.

For a non-empty subset I of L, I is called a *down set* (*up set*) if for $a \in I$ and $x \le a$ $(x \ge a)$ imply $x \in I$. Moreover I is called an *ideal* if it is a down set and $a \vee b \in I$ for all $a, b \in L$. Similarly, F is called a *filter* of L if it is an up set and for $a, b \in F$, $a \wedge b \in F$. F is called a *maximal filter* if for any filter $M \supseteq F$ implies either $M = F$ or $M = L$. A proper ideal (down set) I is called a *prime ideal (down set)* if for $a, b \in L$, $a \wedge b \in I$ imply either $a \in I$ *or* $b \in I$. Similarly, a proper filter (up set) Q is called a *prime filter* (*up set*) if $a \vee b \in Q$ ($a, b \in L$) implies either $a \in Q$ or $b \in Q$. A prime down set P is called a *minimal prime down set* if it does not contain any other prime down set. It is very easy to check that *F* is a filter of L if and only if *L-F* is a prime down set. Moreover, *F* is a prime filter if and only if *L-F* is a prime ideal.

Using Zorn's lemma, [3] have proved the following result:

Lemma 1. *Every proper filter of a lattice with 0 is contained in a maximal filter.*

Similarly we can prove the following results.

Lemma 2. *Every proper ideal of a lattice with 1 is contained in a maximal ideal.*

Following well known result is very easy to prove and we prefer to omit the proof.

Lemma 3. *A subset I of a lattice L is an ideal if and only if $L - I$ is a prime up set. Moreover, I is a prime ideal if and only if $L - I$ is a prime filter.*

Lemma 4. *Let L be a lattice with, F be a filter of L. Then an ideal M of L disjoint from F is a maximal ideal disjoint to F if and only if for any element $a \notin M$ there exists an element $b \in M$ with $a \vee b \in F$.*

Proof. Suppose *M* is maximal and disjoint to *F and* $a \notin M$. Suppose $a \vee b \notin F$ for all $b \in M$. Consider $M_1 = \{ y \in S \mid y \le a \vee b, \text{for some } b \in M \}$. Clearly M_1 is an ideal disjoint to F and . for every $b \in M$ we have $b \le a \vee b$ implies $b \in M_1$. Thus $M \subseteq M_1$. Now $a \notin M$ but $a \in M_1$ imply $M \subset M_1$, which contradicts the maximality of M. Hence there must exist some $b \in M$ such that $a \vee b \in F$.

Conversely, if the ideal M is not a maximal ideal disjoint from F, then there exists a maximal ideal *N* such that $M \subset N$ and $N \cap F = \phi$. For any element $a \in N - M$ there exists an element $b \in M$ such that $a \vee b \in F$. Hence $a, b \in N$

implies $a \vee b \in N$ and so $F \cap N \neq \phi$, which is a contradiction. Thus M must be a maximal ideal disjoint to *F*.

In a lattice L with 0, we define $A^{\perp} = \{x \in L \mid x \wedge a = 0 \ for\ all\ a \in A\}$. This is clearly a down set. Following result gives some characterizations of 0-distributive lattices which is due to [1, 3, 5].

Theorem 5. *Let L be a lattice with 0. Then the following conditions are equivalent.*

(i) L is 0-distributive.

(ii) For all $a \in L, \{a\}^{\perp}$ is an ideal.

(iii) For $A \subseteq L, A^{\perp}$ is an ideal.

(iv) Every maximal filter is prime .

(v) every minimal prime down set of L is a minimal prime ideal.

(vi) Every proper filter of L is disjoint from a minimal prime ideal.

(vii) Each non-zero element $a \in L$ is contained in a prime filter.

(viii) $I(L)$,the lattice of all ideals of L is pseudo complemented.

(ix) I(L) is 0-distributive.

If $\{a\}^{\perp}$ is an ideal, then we denote it by $(a]^{*}$. Similarly if $A^{\perp}$ is an ideal, then we denote it by A^{*} .

Now suppose L is a lattice with 1. For a subset A of L, we define $A^{\perp^d} = \{x \in L \mid x \vee a = 1 \ for\ all\ a \in A\}$. If L is distributive , then $A^{\perp^d}$ is a filter. For $a \in L$, $\{a\}^{\perp^d} = \{x \in L : x \vee a = 1\}$. Moreover, $A^{\perp^d} = \bigcap_{a \in A} \{a\}^{\perp^d}$.
Now we include some characterizations of a 1-distributive lattices which are simply dual to Theorem 5.

Theorem 6. *Let L be a lattice with 1. Then the following conditions are equivalent.*

(i) L is 1-distributive.

(ii) $A^{\perp^d}$ is a *filter* for all $A \subseteq L$.

(iii) $\{a\}^{\perp^d}$ is a *filter* for $a \in L$.

(iv) *Every maximal ideal is prime.*

(v) $F(L)$, *the lattice of all filters is pseudo complemented.*

(vi) $F(L)$ *is 0-distributive.*

Now we include few more characterizations of 1-distributive lattices.

Theorem 7. *Let L be a lattice with 1. Then the following conditions are equivalent.*

(i) L *is 1-distributive.*

(ii) *Every maximal ideal is prime*

(iii) *Every minimal prime up set of L is a minimal prime filter.*

(iv) Every proper ideal of L is disjoint from a minimal prime filter.
(v) Each $a \neq 1$ of L is contained in a prime ideal .

Proof. (i) ⇒(ii) follows from theorem 6.
(ii) ⇒(iii). Let F be a minimal prime up set. Then by lemma 3, $L-F$ is a maximal ideal. Thus by (ii), $L-F$ is a prime (maximal) ideal. Hence by lemma 3, F is a minimal prime filter.
(iii) ⇒(iv). Let I be a proper ideal of L. Then by corollary 2, I is contained in a maximal ideal M. But by lemma 3, $L-M$ is a minimal prime up set. Hence by (iii) $L-M$ is a minimal prime filter disjoint from I.
(iv) ⇒(v). Since $a \neq 1$, so $(a]$ is a proper ideal. So by (iv) there exists a minimal prime filter F disjoint from $(a]$. Thus $(a]$ is contained in $L-I$, which is a prime ideal by Lemma 3.
(v) ⇒(i). Let $a,b,c \in L$ with $a \vee b = 1 = a \vee c$. If $a \vee (b \wedge c) \neq 1$, then there is a prime ideal P such that $a \vee (b \wedge c) \in P$. This implies $a \in P$ and $b \wedge c \in P$. Since P is prime, so $b \in P$ or $c \in P$. Thus either $a \vee b \in P$ or $a \vee c \in P$. This implies $1 \in P$, which is a contradiction. Therefore, $a \vee (b \wedge c) = 1$, and so L is 1-distributive.

Prime separation theorem is a well known result in distributive lattices. Now we give some kind of generalization of that result for 1-distributive lattices in terms of dual annihilators.

Theorem 8. *Let L be a lattice with 1. L is 1-distributive if and only if for any ideal I disjoint with $\{x\}^{\perp^d}$ $(x \in L)$, there exists a prime ideal containing I and disjoint with $\{x\}^{\perp^d}$.*

Proof. Let L be 1-distributive. Consider the set F of all ideals of L containing I and disjoint with $\{x\}^{\perp^d}$. Clearly F is non-empty as $I \in$ F. Then using Zorn's lemma, there exists a maximal element P in F . Now we claim that $x \in P$. If not, then $P \vee (x] \supset P$. So by the maximality of P, $\{P \vee (x]\} \cap \{x\}^{\perp^d} \neq \phi$. Then there exists $t \in P \vee (x]$ and $t \in \{x\}^{\perp^d}$. Then $t \leq p \vee x$ for some that $P \cap \{x\}^{\perp^d} = \phi$ $p \in P$ and $t \vee x = 1$. Thus, $1 = t \vee x \leq p \vee x,$ and so $p \vee x = 1$. This implies $p \in \{x\}^{\perp^d}$, which contradicts the fact. Therefore $x \in P$. Finally, let $z \notin P$. Then $\{P \vee (z]\} \cap \{x\}^{\perp^d} \neq \phi$. Let $y \in \{P \vee (z]\} \cap \{x\}^{\perp^d}$. Then $y \vee x = 1$ and $y \leq p \vee z$ for some $p \in P$. Thus $1 = y \vee x \leq p \vee x \vee z$, which implies $p \vee x \vee z = 1$. Now $x \in P$ implies $p \vee x \in P,$ and $z \vee (p \vee x) = 1$. Hence by Lemma 4, P is a maximal ideal of L, and so by Theorem 6, P is prime.

Conversely, let $x \vee y = 1 = x \vee z$. If $x \vee (y \wedge z) \neq 1$. Then $y \wedge z \notin \{x\}^{\perp^d}$. Thus $[(y \vee z]) \cap \{x\}^{\perp^d} = \phi$. So, there exists a prime ideal P containing] and disjoint with $\{x\}^{\perp^d}$. As $y, z \in \{x\}^{\perp^d}$, so $y, z \notin P$. Thus $y \wedge z \notin P$, as P is prime. This implies $[(y \vee z)] \not\subset P$, a contradiction. Hence $x \vee (y \wedge z) = 1$ and so L is 1-distributive.

Let L be a lattice with 0 and 1. L is called a *Boolean lattice* if it is distributive and complemented. By [2, Theorem 22, p-76], we know that a bounded lattice is Boolean if and only if its prime ideals are unordered. Now we generalize this result for a non distributive bounded lattice.

Theorem 9. *Suppose L is a bounded 0-distributive and 1-distributive lattice. Then L is complemented if and only if all the prime ideals of L are unordered.*

Proof. Suppose L is complemented. Suppose the prime ideals are not unordered. Then there exist prime ideals P, Q such that $P \subset Q$. Let $a \in Q - P$. Since L is complemented, so there exists $a' \in L$ such that $a \wedge a' = 0$ and $a \vee a' = 0$. Now $a \wedge a' = 0 \in P$ and $a \notin P$ imply $a' \in P \subset Q$. This implies $a \vee a' = 1 \in P$, which is a contradiction. Therefore prime ideals are unordered.

Conversely, let the prime ideals are unordered. If L is not complemented then there exists $a \in L$ which has no complement. Set $D = \{x \in L : a \vee x = 1\}$. Since L is 1-distributive, so D is a filter. Set $D_1 = D \vee [a)$. Now $0 \notin D_1$. For if $0 \in D_1$, then $0 \geq d \wedge a$ for some $d \in D$. This implies d is the complement of a, which gives a contradiction. Then by Corollary 2, there exists a maximal filter $F \supseteq D_1$. Since L is

0-distributive, so by Theorem 5, F is prime. Thus $P = L - F$ is a prime ideal disjoint to D_1. Note that $1 \notin (a] \vee P$. For otherwise, $1 = a \vee p$ or some $p \in P$, contradicting $P \cap D = \varphi$. Since L is 1-distributive, so by Theorem 7 (v), there exists a prime ideal Q containing $(a] \vee P$. It follows that, which is impossible as the prime ideals are unordered.

A lattice L with 0 is called 0-*modular* if for all $x, y, z \in L$ with $z \leq x$ and $y \wedge z = 0$ imply $x \wedge (y \vee z) = z$. By [5] we know that a lattice is 0-modular if and only if it does not contain a pentagonal sublattice including 0. Of course every modular lattice with 0 is 0-modular.

A lattice L with 0 is called *weakly complemented* if for any pair a,b of distinct elements of L there exists an element $c \in L$ disjoint from one of these elements but not from the other.

We conclude the paper with the following result.

Theorem 10. *Let L be a complemented and 0-distributive lattice. Then the following conditions are equivalent.*

i) L is 0-modular
ii) L is unicomplemented
iii) L is Boolean
iv) L is weakly complemented

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Let L be not unicomplemented. Then there exists $a \in L$ which has two complements b and c. Then $a \wedge b = a \wedge c = 0$ and $a \vee b = a \vee c = 1$. If b and c are not comparable, then $\{0, a, b, c, 1\}$ forms a diamond, which implies L is not 0-distributive and this gives a contradiction. Again if $b < c$, then $\{0, a, b, c, 1\}$ forms a pentagonal sublattice which contradicts the 0-modularity of L. Therefore L must be unicomplemented.
(ii) $\Rightarrow$ (iii) holds by [5, Corollary 2.3].
(iii) $\Rightarrow$ (i) is trivial as every Boolean lattice is distributive, and so it is 0-modular.
(iii) $\Leftrightarrow$ (iv). If L is Boolean, then it is sectionally complemented, and so it is weakly complemented. On the other hand, if L is weakly complemented, then by [5, Corollary 2.2], L is Boolean.

Theorem 11. *Let L be a 0-distributive lattice and* $[0, x]$ *is 1-distributive for each* $x \in L$. *Then the following conditions are equivalent:*
(i) $[0, x]$ *is complemented for each* $x \in L$.
(ii) $(x] \vee (x]^* = L$ *for each* $x \in L$.
(iii) The prime ideals of $[0, x]$ *are unordered.*
Proof. (i) $\Rightarrow$ (ii)
Let $x \in L$ choose any $y \in L$
Now, $0 \le y \wedge x \le y$ since $[0, y]$ is complemented.
So, there exists $t \in [0, y]$ such that $y \wedge x \wedge t = 0, (y \wedge x) \vee t = y$ which implies $y \wedge t \wedge x = 0$ and so $t \wedge x = 0$. This implies $t \in (x]^*$.
Thus, $y = (y \wedge x) \vee t \in (x] \vee (x]^*$.

Hence $(x] \vee (x]^* = L$.
(ii) $\Rightarrow$ (iii)

Suppose (ii) holds. Let the ideals of $[0,x]$ for some $x \in L$ are not unordered. Then there exists prime ideals P_1, Q_1 of $[0,x]$ such that $P_1 \subset Q_1$ then there exists $y \in Q_1 - P_1$

Since Q_1 is prime, so, $x \notin Q_1$.

By (ii) $(y] \vee (y]^* = L$. Then $x \in (y] \vee (y]^*$. This implies $x \leq p \vee q$ for some $p \in (y], q \in (y]^*$ so $q \wedge y = 0 \in P$. But $y \notin P$. So $q \in P$ as P is prime and so $q \in Q$. Also, $p \leq y \in Q$.

Hence $x \leq p \vee q$ implies $x \in Q$ which gives a contradiction. So (iii) holds.

(iii)$\Rightarrow$(i) follows from Theorem 9.

Semi prime filter.

Recently Y. Rav in [4] have given the concept of semi prime filter to generalize the idea of 1-distributive lattices. Let F be a filter of a lattice L . F is called a semi prime filter if for all $x, y, z \in L$, $x \vee y \in F$ and $x \vee z \in F$ imply $x \vee (y \wedge z) \in F$. Every filter of a distributive lattice is semi prime. Moreover, every prime filter is semi prime. In any lattice *L, L* itself is a semi prime filter.

In pentagonal lattice $\{0,a,b,c,1; a < b, a \wedge c = b \wedge c = 0, a \vee c = b \vee c = 1\}$ all the ideals except $[b)$ are semi prime. Again in the diamond lattice $\{0,a,b,c,1; a \wedge b = b \wedge c = a \wedge c = 0, a \vee b = b \vee c = a \vee c = 1\}$ all the filters except L are not semi prime.

Theorem 12. Let L be a lattice with 1. For any $A \subseteq L$, $A^{\perp^d}$ is a semi prime filter if and only if L is 1-distributive.

Proof. By Theorem-6 we need only to prove that in a 1-distributive lattice $A^{\perp^d}$ is semi prime. Let $x \vee y \in A^{\perp^d}$ and $x \vee z \in A^{\perp^d}$. Then

$x \vee y \vee a = 1 = x \vee z \vee a \forall a \in A$ .Since L is 1-distributive, so

$(x \vee a) \vee (y \wedge z) = 1$.

Thus $[x \vee (y \wedge z)] \vee a = 1$ for each $a \in A$. This implies $x \vee (y \wedge z) \in A^{\perp^d}$,and so $A^{\perp^d}$ is 1-distributive.

Theorem13. *Let L be a lattice with 1. Then any maximal ideal of L disjoint from a semi prime filter is prime.*

Proof: Let F be a semi prime filter of *L*. Suppose *P* is a maximal ideal such that $F \cap P = \phi$. Then $L - P$ is a maximal prime up set containing *F.* Suppose $x, y \in L - P$. Then by Lemma 4, there exist $a, b \in P$ such that $a \vee x \in F, b \vee y \in F$. This implies $a \vee b \vee x \in F$ and $a \vee b \vee y \in F$. Since F is semi prime, so $(a \vee b) \vee (x \wedge y) \in F \subseteq L - P$. Since $L - P$ is prime and $a \vee b \in P$, so $x \wedge y \in L - P$.

Therefore, $L - P$ is a filter and so *P* is a prime ideal.

We conclude the paper with the following separation theorem.

Theorem 14. *Let L be a lattice with 1 and I be a ideal such that* $I \cap A^{\perp^d} = \phi$ *for some non empty subset A of L. Then L is 1-distributive if and only if there exists a prime ideal P containing I such that* $P \cap A^{\perp^d} = \phi$.

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