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A Note on Star Critical Ramsey (C_n,K_6) Numbers for Large *n*

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Abstract. The study of Ramsey theory was initiated by the paper on a problem of formal logic written Ramsey. Let K_n denote the complete graph on n vertices. For any red/blue colouring of K_n, let H_R and H_B denote the red and blue subgraphs of K_n respectively so that $K_n = H_R \bigoplus H_B$. Let H, G be simple graphs. If there exists a red copy H in H_R or a blue copy G in H_B, we say that $K_n \longrightarrow (H, G)$. One branch of Ramsey theory, deals with the exact determination of Ramsey number, $r(H, G)$, defined as the smallest positive integer n such that $K_n \longrightarrow (H, G)$. For small size graphs H and G, Ramsey number $r(H, G)$ has been studied extensively in the last five decades. In the special case $H = G = K_n$ the exact determination of $r(K_n, K_n)$, swifts expeditiously from the apparent $r(K_3, K_3) = 6$, to the unmanageable $r(K_5, K_5)$. Currently, the best known lower and upper bounds for $r(K_5, K_5)$ are 43 and 48 ([7,8]). A closely related recent development in this area of study is the determination of Star critical Ramsey number $r^*(H, G)$ defined as the largest integer k such that $K_{r(G,H)-1} \sqcup K_{\{1,k\}} \longrightarrow (H, G)$. In this work, we find $r^*(C_n, K_6)$ when $n \geq 10$.

Keywords: Graph theory, Ramsey theory, Ramsey critical graphs

AMS Mathematics Subject Classification (2010): 05C55, 05C38, 05D10

1. Introduction

After the introduction of Ramsey Theory, many authors have investigated how to extend the initial result of Erdös and Szekeres, namely $r(K_3, K_3) = 6$. However, even after a century, very little has been done despite the use of 'state of the art' computers. This sentiment has been eloquently expressed by the great mathematician Paul Erdös as exemplified by the following quotation (see 1990 Scientific American article by Ronald Graham and Joel Spencer).

'Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five $r(K_5, K_5)$. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six $r(K_6, K_6)$, however, we would have no choice but to launch a pre-emptive attack'.

Paul Erdös

(1990 Scientific American article by Ronald Graham & Joel Spencer)

One other branch of mathematics that emerged subsequently is the determination of $r(C_n, K_m)$. In fact it has been conjectured by Bondy and Erdös in 1978 that $r(C_n, K_m) = (n-1)(m-1)+1$ for all $n \geq m \geq 3$, with the exception of $n = m = 3$ ([1]). However, according to the current survey papers on Ramsey theory, the best-known cycle-complete Ramsey numbers related to $r(C_n, K_m)$ are currently known for $n \geq 4m+2,\ m \geq 3$ (see [8]).

The fact that, Star critical Ramsey numbers [3], give a more detailed insight into understanding the behaviour of corresponding Ramsey numbers has triggered an interest on finding $r^*(C_n, K_m)$. Working along this line, in this paper, we concentrate on the special case of $r^*(C_n, K_6)$ for large values of n ($n \geq 15$).

2. Terminology

The total number of edges of a complete graph on n vertices will be denoted by $|E(K_n)|$ so that $|E(K_n)| = n(n-1)/2$. The independence number, which is defined as the size of the largest independent set of a graph will be denoted by $\alpha(G)$. Given a vertex $v \in V(G)$ of a graph G, the neighbourhood of v in G that is defined as the set of vertices adjacent to v in G will be denoted by $\Gamma(v)$. The degree of a vertex v in G, denoted by $d(v)$ is defined as the cardinality of $|\Gamma(v)|$, where $\Gamma(v)$ represents the set of vertices adjacent to v. The minimum degree of a graph $G = (V, E)$ denoted by $\delta(G)$, is defined as the minimum of the degrees of the vertices of G. Similarly, the maximum degree of a graph $G = (V, E)$ denoted by $\Delta(G)$, is defined as the maximum of the degrees of the vertices of G. Furthermore, we say that a graph is a r regular graph (or simply regular), if degree of each vertex of graph G is equal to r. (i.e., $\delta(G) = \Delta(G) = r$). Given a graph G and a non-empty subset S of $V(G)$, the induced subgraph of S in G denoted by $G[S]$ is defined as the subgraph obtained by deleting all the vertices of the complement of S from G. For a graph G and two disjoint subgraphs H and H' of G, we denote the set of edges between H and H' by $E(H, H')$.

3. Methodology

We use three lemmas to calculate $r^*(C_n, K_6)$ when $n \geq 15$. The Lemmas we use are already proven results and relevant references are indicated in parentheses.

Lemma 1. ([5], Lemma 2) *A C_n - free graph G of order N with independent number less than or equal to m has minimal degree greater than or equal to $N - r(C_n, K_m)$.*

Lemma 2. ([4], Lemma 8) *A C_n - free graph (where $n \geq 15$) of order* $5(n-1)$ *with no independent set of size 6 contains a $5K_{n-1}$.*

Lemma 3. ([2], Lemma 5)

Suppose G contains the cycle $(u_1, u_2, \ldots u_{n-1}, u_1)$ of length $n-1$ but no cycle of length n. Let $Y = V(G) \setminus \{u_1, u_2, \ldots u_{n-1}\}$. Then,

(a) *No vertex* $x \in Y$ *is adjacent to two consecutive vertices on the cycle.*
(b) *If* $x \in Y$ *is adjacent to* u_i *and* u_j *then* $u_{i+1}u_{j+1} \notin E(G)$.
(c) *If* $x \in Y$ *is adjacent to* u_i *and* u_j *then no vertex* $x \in Y$ *is adjacent to both* u_{i+1} *and* u_{j+2}.
(d) *Suppose* $\alpha(G) \leq m-1$ *where* $m \leq (n+2)/2$ *and* $\{x_1, x_2, \dots, x_{m-1}\} \subseteq Y$ *is an* $(m-1)$ *- element independent set. Then, no member of this set is adjacent to* $m-2$ *or more vertices on the cycle (We have taken the liberty of making a slight correction to the inequality* $m \leq (n+2)/2$ *of the original [2], Lemma 5(d)).*

4. Main result

Here we provide a theorem with proof to find $r^*(C_n, K_6)$.

Theorem 1. *If* $n \geq 15$ *then* $r^*(C_n, K_6) = 4n-2$.

Proof: By Lemma 1 and Lemma 2 (see [6,9]), $r(C_n, K_6) = 5(n-1)+1 = 5n-4$. To find a lower bound for $r^*(C_n, K_6)$, colour the graph $K_{5n-4} \backslash K_{1,n-2}$, such that the red graph consists of a $5K_{n-1} \sqcup K_{1,1}$ as illustrated in Figure 3, where a thick dotted line represents blue edges bundle (containing $(n-1)^2$ or $(n-1)$ number of edges) except and a thin solid line represents a single red edge. Hence, $K_{5n-4} \backslash K_{1,n-2} \nrightarrow (C_n, K_m)$. Therefore, $r^*(C_n, K_6) \geq 4n-2$. To show that $r^*(C_n, K_6) \leq 4n-2$, assume that there exists a red C_n- free red/blue coloring of a graph $G = K_{5n-4} \backslash K_{1,n-3}$ that contains no blue K_6. Let H be the graph obtained by deleting the vertex of degree $4n-2$ (say v) from G (i.e., $H = G \backslash v$). Since G is a graph on $5n-4$ vertices, H is a graph on $5(n-1)$ vertices containing no red C_n or a blue K_6. Therefore, by Lemma 4, H_R contains a red $5K_{n-1}$. Let $V_1, V_2, \dots V_5$ denote the vertex sets of these five K_{n-1} components. In order to avoid a red C_n, v can be adjacent in red to at most one red neighbor in each of the 5 sets $V_1, V_2, \dots V_5$. We exercise the prerogative of assuming that $V_1, V_2, \dots, V_5$ represent in the descending order with respect to the number of vertices connected to v (disregarding the color). For each $i \in \{1, 2, \dots, m-1\}$, let U_i represents the number of vertices of V_i not connected to v.

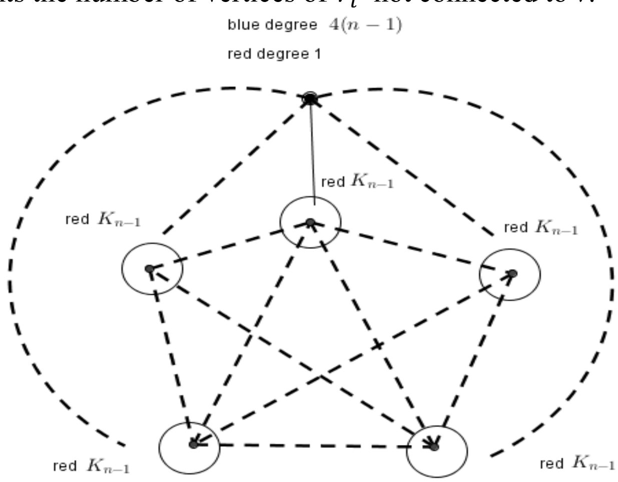

Figure 1: A red C_n- free colouring of $K_{5n-4} \backslash K_{1,n-2}$ with no blue K_6

Claim 2. $|U_5| \le n-3$ and $|U_4| \le n-7$.

Proof of Claim 2. $|U_1| + \cdots + |U_5| \le$ n-3 together with $0 \le |U_1| \le |U_2| \le |U_3| \le |U_4| \le |U_5| \le n-3$ gives $|U_5| \le n-3$.

To prove $|U_4| \le n-7$, on the contrary assume that $|U_4| \ge n-6$. Then,

$$n-3 = |U_1| + |U_2| + |U_3| + |U_4| + |U_5| \ge |U_4+U_5| \ge 2|U_4| \ge 2(n-6).$$

That is $n \le 9$, a contradiction as $n \ge 10$. Hence, the claim.

Next continue with the proof of Theorem 1. By Claim 2, as $|U_5| \le n-3$, we get V_5 has at least 2 vertices adjacent to v. In order to avoid a red C_n, at least one of these two vertices have to be adjacent to v in blue. That is, there exists $v_5 \in V_5$ such that (v, v_5) is blue. Next, V_4 has at least 6 vertices connected to v. Out of these 6 vertices, at most two vertices can be connected to v in red. Therefore, there exists a vertex $v_4 \in V_4$ adjacent to $\{v_5, v\}$, in blue. That is $\{v_4, v_5, v\}$, induces a blue K_3. Next, we get that there is a vertex $v_3 \in V_3$ such that $\{v_3, \ldots, v_5, v\}$ induces a blue K_4. Arguing in the same manner, we conclude that there exists $v_i \in V_i$ $(1 \le i \le 5)$ such that v_1 is adjacent in blue to all the vertices of $\{v_2, \ldots, v_5, v\}$, where $\{v_2, \ldots, v_5, v$, induces a blue K_5. Therefore, $\{v_1, v_2, \ldots, v_5, v\}$, will induce a blue K_6, a contradiction.

5. Discussion

First of all, the authors would like to thank the reviewers for their useful comments.
From Theorem 1 obtained for $m = 6$, we envisage the following generalisation.

Result. *A C_n-free graph, where $n \ge (m-3)(m-1)$ with $m \ge 7$, of order $(n-1)(m-1)$ with no independent set of order m contains an copy of $(m-1)\, K_{n-1}$. Moreover, for $n \ge (m-3)(m-1)$ with $m \ge 7$, $r^*(C_n, K_m) = (m-2)(n-1)+2$.*

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