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Chebychev Subspaces of Orlicz Function Space

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Abstract. In this paper it is proved that G is a Chebychev subspace of a Banach space X if and only if $L^{\phi}(\mu, G)$ is a Chebyshev subspace of $L^{\phi}(\mu, X)$, where $L^{\phi}(\mu, X)$ is an Orlicz function space with Luxemburg norm.

Keywords: Proximinally additive, Chebychev space, Orlicz space.

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1. Introduction

Let ϕ be an Orlicz function on $[0,\infty)$ (i.e. a continuous, strictly increasing convex function satisfying $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$). Let (Ω, μ) be a finite measure, for a real Banach space X, The orlicz space $L^{\phi}(\mu, X)$ is a space of all measurable functions $f: \Omega \longrightarrow X$ such that $\int_{\Omega} \phi(c^{-1}\|f(t)\|)\, d\mu(t) < \infty$, for some $c > 0$. Define a Luxemburg norm on $L^{\phi}(\mu, X)$ by $\|f\|_{\phi} = \inf\left\{c > 0: \int_{\Omega} \phi(c^{-1}\|f(t)\|)\, d\mu(t) \leq 1\right\}$, for more see [9].

For the subset G of the normed linear space $(X, \|.\|)$, we define, for $x \in X$, $d\,(x\,, G) = \inf\{\|x - g\|: g \in G\}$. If G is a subspace of X, an element $g_0 \in G$ is called a best approximant of x in G if $\|x - g_0\| = d(x, G)$. We shall denote the set of all best approximants of x in G as $P(x\,, G)$. If for each $x \in X$, the set $P\,(x, G) \neq \emptyset$, then G is said to be proximinal in X, and if $P(x, G)$ is a singleton for each $x \in X$ than G is called a Chebychev subspace. A subspace G of a Banach space X is said to be proximinally additive if G is closed and $z_1 + z_2 \in P(x_1 + x_2, G)$ whenever $z_1 \in P(x_1, G)$and $z_2 \in P(x_2, G)$. For more see [6], [7].

In this paper we prove that: G is a Chebychev subspace of a Banach space X if and only if $L^{\phi}(\mu, G)$ is a Chebyshev subspace of $L^{\phi}(\mu, X)$, where $L^{\phi}(\mu, X)$ is an Orlicz function space with Luxemburg norm. In [3], it was proved the same result for the linear metric space $L^{\phi}(\mu, X)$, with ϕ is a modulus function, and $d(f, g) = \int_{\Omega} \phi(\|f(t) - g(t)\|) d\mu(t)$ is the metric, for more results about this metric space see [2, 3,4,5].

Throughout this paper we take $\Omega = [0,1]$ and ϕ is an Orlicz function with $\phi(0) = 0, \phi(1) = 1$.

2. Main results

Lemma 2.1. [3] Suppose that G is a proximinal subspace of a Banach space X and that G is proximinally additive in X. Then G is Chebyshev.

Theorem 2.2. Let X be a Banach space and G be a closed subspace of X. if G is proximinally additive in X, then $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$.

Proof: Suppose $g_1 \in P(f_1, L^{\phi}(\mu, G))$, and $g_2 \in P(f_2, L^{\phi}(\mu, G))$, then by [8, corollary 2.1] we get $g_1(t) \in P(f_1(t), G)$, $g_2(t) \in P(f_2(t), G), \forall t$.

Hence, $\forall t \in \Omega, \forall y \in G$

$d\big((f_1 + f_2)(t), G\big) = \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \leq \|(f_1 + f_2)(t) - y\|$

Thus, $\forall t \in \Omega, \forall y \in L^{\phi}(\mu, G) : \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \leq \|(f_1 + f_2)(t) - h(t)\|$.

Since ϕ is increasing, then $\forall h \in L^{\phi}(\mu, G)$ we have that

$\int_0^1 \phi(c^{-1}\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|)\, d\mu(t) \leq \int_0^1 \phi(c^{-1}\|(f_1 + f_2)(t) - h(t)\|\, d\mu(t))$

Thus,

$$\inf\{c > 0 \colon \int_0^1 \phi(c^{-1}\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|)\, d\mu(t)\ \}$$

$$\leq \inf\{c > 0 \colon \int_0^1 \phi(c^{-1}\|(f_1 + f_2)(t)h(t)\|\, d\mu(t))\}$$

Therefore, $\forall h \in L^{\phi}(\mu, G)$, $\|(f_1 + f_2) - (g_1 + g_2)\|_{\phi} \leq \|(f_1 + f_2) - \square\|_{\phi}$

And so, $g_1 + g_2 \in\ P(f_1 + f_2, L^{\phi}(\mu, G))$. Thus, $L^{\phi}(\mu, G)$ is proximinally additive. ■

Theorem 2.3. Let X be a Banach space and G be a closed subspace of X. If $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$, then G is proximinally additive in X.

Proof: Suppose $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$, and let $z_i \in P(x_i, G)$ for $i = 1, 2$; we want to show $z_1 + z_2 \in P(x_1 + x_2, G)$.

Now let $f_i(t) = x_i$ and $g_i(t) = z_i, \forall t$ and for $i = 1,2$.

It is clear such that $f_i, g_i \in L^{\phi}(\mu, X)$, and $g_i \in L^{\phi}(\mu, G)$, $i = 1,2$,.

First, we show that $g_i \in P(\ f_i, L^{\phi}(\mu, G))$, $i\ =\ 1, 2$.

Now, for $i\ = 1, 2$, we have

$z_i \in P(x_i, G) \Longrightarrow \|x_i - z_i\| \leq \|x_i - y\|, \forall y \in G$

$\Longrightarrow \|\ f_i(t) - g_i(t)\| \leq \|\ f_i(t) - y\|, \forall y \in G, \forall t$

$\Longrightarrow \|\ f_i(t) - g_i(t)\| \leq \|\ f_i(t) - h(t)\|, \forall t, \forall h \in L^{\phi}(\mu, G)$

Since ϕ is strictly increasing, then $\forall h \in L^{\phi}(\mu, G)$ we have that

$\int_0^1 \phi(c^{-1}\|f_i(t) - g_i(t)\|)\, d\mu(t) \leq \int_0^1 \phi(c^{-1}\|f_i(t) - h(t)\|\, d\mu(t))$

Hence, $\forall h \in L^{\phi}(\mu, G)$:

$$\inf\{c > 0 \colon \int_0^1 \phi(c^{-1}\|f_i(t) - g_i(t)\|)\, d\mu(t) \leq 1\}$$

$$\leq \inf\{c > 0 \colon \int_0^1 \phi(c^{-1}\|f_i(t) - h(t)\|)\, d\mu(t) \leq 1\}$$

Thus, $\|f_i - g_i\|_{\phi}\ \leq \|f_i - h\|, i = 1,2, \forall h \in L^{\phi}(\mu, G)$.

So we get that $g_i \in P(\ f_i, L^{\phi}(\mu, G))$, $i\ =\ 1, 2$.

Since $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$, then

$$g_1 + g_2 \in\ P(f_1 + f_2, L^{\phi}(\mu, G)).$$

By corollary 2.1 of [8], we have that

$$g_1(t) + g_2(t) \in P(f_1(t) + f_2(t), G).$$

Hence, $z_1 + z_2 \in P(x_1 + x_2, G)$. Therefore G is proximinally additive in X. ■

Corollary 2.4. Let X be a Banach space and G be a closed subspace of X. $L^{\phi}(\mu, G)$ is proximinally additive in $L^{\phi}(\mu, X)$ if and only if G is proximinally additive in X.

Theorem 2.5. Let G be a closed subspace of a Banach space X. Then the following are equivalent :

(i) G is proximinal in X

(ii) $L^{\phi}(\mu, G)$ is proximinal in $L^{\phi}(\mu, X)$.

Proof: By[1, theorem 3.7] and [8, theorem 2.1] we get that:

G is proximinal $\Leftrightarrow L^1(\mu, G)$ is proximinal $\Leftrightarrow L^{\phi}(\mu, G)$ is proximinal . ■

Theorem 2.6. Let G be a closed subspace of a Banach space X which is proximinally additive in X, then the following are equivalent:

(i) G is a Chebyshev subspace of X.

(ii) $L^{\phi}(\mu, G)$ is a Chebyshev subspace of $L^{\phi}(\mu, X)$.

Proof: By theorem (2.5), and corollary 2.4 ■

Orlicz spaces generalize $L^p(\mu, X)$ for $1 < p < \infty$, in the sense that if $\phi(t) = t^p$, then $\|f\|_{\phi} = \|f\|_p$ and so $L^{\phi}(\mu, X) = L^p(\mu, X)$. Hence, the following theorem from [3] is a direct result of corollary 2.4:

Theorem 2.7. Let G be a closed subspace of a Banach space X. Then the following are equivalent :

(i) G is proximinally additive in X.

(ii) $L^p(\mu, G)$ is proximinally additive in $L^p(\mu, X)$, $1 < p < \infty$.

3. Conclusion

If G is a closed subspace of a Banach space X, then we get that: G is a Chebychev subspace of a Banach space X if and only if $L^{\phi}(\mu, G)$ is a Chebyshev subspace of $L^{\phi}(\mu, X)$, where $L^{\phi}(\mu, X)$ is an Orlicz function space with Luxemburg norm.

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