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On the Hamiltonian for Isothermal Compressible Viscous Fluids

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Abstract. This article defines a Bernoulli equation for isothermal compressible viscous fluids. The Bernoulli equation includes the use of variable density, an isothermal potential modification to be added to the pressure/force potential, and the standard definition of the dissipation function was modified. Since the standard definition of the dissipation function is *inconsistent* with the development of the Hamiltonian dynamical method in order to obtain the compressible Navier Stokes momentum equations for isothermal viscous compressible fluids. The Navier Stokes momentum equations for an isothermal viscous compressible fluid are obtained using Hamilton's equations where the fluid's Hamiltonian is defined from the compressional viscous fluid Bernoulli equation. The continuity equations of the viscous fluid was also obtained by optimization with respect to time of the density-Jacobean product since the fluid parcel motion does not change the mass of the parcel in time.

Keywords: Bernoulli's Law, isothermal compressible viscous fluids, Hamiltonian dynamical methods

AMS Mathematics Subject Classification (2010): 76N99

1. Introduction

This article defines a Bernoulli equation for compressible viscous fluid with initial time density, $\rho_o = \rho(t_o, \vec{x}_o) = \rho(t_o, \vec{x}(t_o))$, where the spatial fluctuations in the potential energy per unit mass occur with constant temperature (isothermal) is based on previous research of the incompressible viscous Bernoulli equation developed in references [2 & 3]. Most top fluid mechanics researchers in the 20th century, for example Professors Lamb [8], Serrin [4], and Batchelor [1], have written elegant seminal research articles or books in fluid mechanics on how to develop the fluid's momentum equations using sophisticated mathematical techniques requiring the fluid's parcels coordinates to be mathematically designated as Eulerian or Lagrangian coordinate variables, but such mathematical *labels or designations* are not *physically* real or warranted. Where as in this article, no effort shall be made to designate the fluid parcel's coordinates as either Eulerian or Lagrangian, since articles [3] and [7] have shown no *physical* difference in coordinate variables (and therefore their velocity) really *exists* in fluid parcels due to these mathematical coordinate designations or labels. Additionally, the partial spatial differentiation with respect to the fixed Cartesian coordinate system, X_i, is labeled for convenience as the parcel's center coordinates, x_i, i.e., $\frac{\partial}{\partial X_i} \equiv \frac{\partial}{\partial x_i}$, i.e. since the fluid parcel does not know the difference between the Eulerian coordinates and Lagrangian coordinates. Please note that the Jacobean determinant is express in terms of the current fluid parcel's center coordinates, x_i, measured from an arbitrary fixed Cartesian coordinate system, X_i, (not shown) and a fixed Cartesian coordinates, Y_i, which are dummy integration variables relative to the center of the fluid parcel's volume integral at time t_o.

The Bernoulli equation for isothermal compressible viscous fluids *defined* in this article is a physical model innovation. Finding the Bernoulli equation for viscous incompressible fluids was very interesting research done in [2,3]. Yet, defining Bernoulli equation for isothermal compressible viscous fluids and showing how to obtain the Navier-Stokes for viscous compressible fluids via Hamilton's dynamical methods is an exciting theoretical physical model which was once thought to be beyond the bounds of what was physically possible, this notion of impossibility might explain the lack of published references in this topic.

The compressible Navier Stokes momentum equations are obtained using Hamilton's equations where the fluid's Hamiltonian is defined as a Bernoulli equation which includes an isothermal potential modification, $RTln(\frac{\rho(t,\vec{x}(t))}{\rho_o})$, where R is the specific gas constant of the ideal gas law [1] multiplied times the fluid temperature, T, is the proportionality constant times the natural logarithm of the ratio of the current density, $\rho = \rho(t, \vec{x}(t))$, to an initial density[1], ρ_o, to be added to the pressure/force potential, $\frac{p(\vec{x}(t))}{\rho(t,\vec{x}(t))} + \phi(\vec{x}(t))$. The compressible viscous Bernoulli equation also uses the variable density instead of a spatially homogeneous numeric constant density, ρ_o, as found in incompressible fluids. The isothermal potential term physically represents changes in volume as seen with the continuity equation, $\rho J = \rho_o$, the Jacobean of the transformation [4], $J = \frac{\partial(x_1(t),x_2(t),x_3(t))}{\partial(Y_1,Y_2,Y_3)}$, which represent the change in volume, as shown below.

$$RTln\left(\frac{\rho}{\rho_o}\right) = -RTln(J),$$

The partial derivative with respect to space of the equation above is given by *(Eq. 1)*,

$$[-\frac{\partial}{\partial x_i} RTln\left(\frac{\rho}{\rho_o}\right)]_T = -\frac{RT}{\rho}\frac{\partial \rho}{\partial x_i} + \frac{RT}{\rho_o}\frac{\partial \rho_o}{\partial x_i} = -\frac{RT}{\rho}\frac{\partial \rho}{\partial x_i} = \frac{RT}{J}\frac{\partial J}{\partial x_i} \tag{1}$$

Where the initial density is either a spatially homogeneous numeric constant or independent of the fixed Cartesian coordinates $\frac{\partial \rho_o}{\partial x_i} = 0$ since the initial coordinates x_{oi} are fixed with respect to the fixed Cartesian coordinates after some time, $t > t_o$. Therefore, it can be seen that for constant temperature the thermal energy represented by RT is changed by the ratio $\frac{(\frac{\partial J}{\partial x_i})}{J}$. Additionally, the standard definition of the dissipation

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[1] The initial density, ρ_o, could also be is a spatially homogeneous numeric constant in some cases, i.e. independent of coordinates and time.

function[2] [1], $D = \frac{1}{\rho}\tau_{k,j}\frac{\partial u_k}{\partial x_j}$, was modified to $D = \tau_{k,j}\frac{\partial}{\partial x_j}(\frac{u_k}{\rho})$, since the standard dissipation function definition is *inconsistent* with the development of the Hamiltonian dynamical method in order to obtain the compressible Navier Stokes momentum equations for viscous fluids. Notice, if the current fluid density is the initial density, ρ_o, and does not change i.e. fluid is incompressible then the natural logarithm of 1 is zero, so this additional potential term due to density changes vanishes, additionally the modified dissipation function becomes the standard dissipation function since the constant density can be taken out of the partial differentiation. The continuity equations of the viscous fluid was also obtained by optimization with respect to time of the density-Jacobean product since the fluid parcel motion does not change the mass of the parcel in time.

2. A definition of a isothermal compressible Bernoulli equation for viscous fluid

The isothermal compressible Bernoulli equation can be *defined* for, time t, and initial time, t_0, as follows in *(Eq. 1)* is a generalization of the definition of the incompressible Bernoulli equation found in reference [3] which was found to be the first integral of the incompressible Navier Stokes momentum equations. The compressible dissipation function definition, $D = \tau_{k,j}\frac{\partial}{\partial x_j}(\frac{u_k}{\rho})$, is different from the standard dissipation function definition, $D = \frac{1}{\rho}\tau_{k,j}\frac{\partial u_k}{\partial x_j}$, found in reference [1] because it uses the variable density inside the integrand's spatial partial differentiation of a component of the fluid velocity as shown in *(Eq. 2)*. Therefore, the first integral of the compressible Navier-Stokes momentum equations is given by *(Eq. 2)*.

$$B(t,\vec{x}(t),\vec{u}(t,\vec{x}(t))|T) = \frac{1}{2}u_k^2\big(t,\vec{x}(t)\big) + \Psi(t,\vec{x}(t)|T) + \int_{t_0}^{t}\tau_{k,j}\frac{\partial}{\partial x_j}(\frac{u_k}{\rho})d\tau + I_o \tag{2}$$

With the *isothermal* restriction that the fluid temperature, T, is being held *fixed*, as denoted by $|T$ in the functional arguments and/or bracket, $]_T$, when performing the spatial partial derivatives in the new potential per unit mass, Ψ, where, I_o is the internal energy per unit mass, $\frac{1}{2}u_k^2\big(t,\vec{x}(t)\big)$ is the kinetic energy per unit mass, and $\Psi(t,\vec{x}(t)|T) = \frac{p(\vec{x}(t))}{\rho(t,\vec{x}(t))} + [RTln(\frac{\rho(t,\vec{x}(t))}{\rho_o})]_T + \phi(\vec{x}(t))$ is the new potential per unit mass based on potential pressure, p, external force potential, ϕ, and the fluid's compression potential, $[RTln\left(\frac{\rho(t,\vec{x}(t))}{\rho_o}\right)]_T$, due to compression or rarefaction of the fluid volume only since the fluid temperature, T, is held fixed by an isothermal process[3]. Notice, if ρ becomes a spatially homogeneous numeric constant density, ρ_o, for all times the isothermal compressible Bernoulli equation for viscous fluids becomes the Bernoulli equation for incompressible fluids [3]. Recall from reference [3], the Bernoulli equation exist in a volume integral, which can be spatially integrated by parts results in *(Eq. 3)* with the understanding the resulting surface integral is zero as it was shown in reference [3], page 174.

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[2] The Einstein summation convention is used throughout this manuscript with indices j and i equal to 1, 2 and 3, unless otherwise noted.

[3] This process could be also isothermal and adiabatic.

$$B(t,\vec{x}(t),\vec{u}(t,\vec{x}(t))|T) = \frac{1}{2}u_k^2\big(t,\vec{x}(t)\big) + \Psi(t,\vec{x}(t)|T) - \int_{t_0}^{t}\frac{1}{\rho}\frac{\partial}{\partial x_j}(\tau_{k,j})u_k\,d\tau + I_o \quad (3)$$

<u>Hamiltonian definition for viscous compressible fluids</u>

The time integral can be transformed into a line integral by the substitution $dx_k = u_k d\tau$ in the Bernoulli equation *(Eq. 3)* and since the Bernoulli formula is equivalent to the energy of the fluid, therefore, then the Bernoulli formula time integral may take the form of a spatial line integral where an explicit sums are used to avoid confusion on the third additive term in *(Eq. 3)* since the index k appears 3 times. The isothermal compressible Bernoulli equation *(Eq. 3)* can be *<u>defined</u>* as the isothermal compressible fluid *Hamiltonian* equation used in the *Hamiltonian* method *(Eq. 4)*. The time derivative of the integrals in (Eq. 3) and (Eq. 4) yield the same results, therefore they represent the same integral since the additive constant is zero.

$$H(t,\vec{x}(t),\vec{u}(t,\vec{x}(t))|T) = \frac{1}{2}u_k^2\big(t,\vec{x}(t)\big) + \Psi(t,\vec{x}(t)|T) - \sum_{k,j=1}^{3}\int_{x_{ok}}^{x_k(t)}\frac{1}{\rho}\frac{\partial}{\partial x_j}\tau_{k,j}\,dx_k + I_o \quad (4)$$

where $x_{ok} = x_k(t_0)$.

Notice the *Hamiltonian* equation of compressible viscous fluids is a constant in time as shown in *(Eq. 5)*. Performing the field or material derivative with respect time of equation *(Eq. 4)* where the constant density cancels out results in the dot product of the fluid velocity components with the Navier-Stokes momentum equations. Since $[-\frac{p}{\rho^2}\frac{\partial\rho}{\partial x_k}u_k + \frac{RT}{\rho}\frac{\partial\rho}{\partial x_k}u_k - \frac{RT}{\rho_o}\frac{\partial\rho_o}{\partial x_k}u_k]_T = 0$, because the last term $\frac{\partial\rho_o}{\partial x_i}$ is zero due to the initial location is being held fixed with respect to the fixed Cartesian coordinates for $t > t_o$, and by the use of the thermo dynamical specific ideal gas law, $RT = \frac{p}{\rho}$, the first two terms cancel out.

$$[\frac{dH}{dt}]_T = u_k(\frac{du_k}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_k} + \frac{\partial\phi}{\partial x_k} - \frac{1}{\rho}\frac{\partial}{\partial x_j}\tau_{k,j}) = 0 \quad (5)$$

The compressional Navier-Stokes momentum equations can be derived using the usual Hamiltonian methods with the Legendre Transformation found in reference [5] with the understanding the temperature is held fixed while performing spatial differentiations in the scalar energy potential per unit mass, Ψ.

Theorem 1. Navier Stokes Momentum equations can be derived from the Hamilton equations

If the initial fluid density is independent of the fixed Cartesian coordinates, $\frac{\partial\rho_o}{\partial x_i} = 0$, the Hamiltonian, H, is given by *(Eq. 4)* and the *Lagrangian* function is defined by as $L(t,x_i,\frac{dx_i}{dt},u_i) = u_i\frac{dx_i}{dt} - [H]_T$, where the Lagrangian is in the volume integrand of the functional, $F[x_i,u_i;t] = \int_{t_o}^{t}\iiint_{V(t_o)} L(\tau,x_i,\frac{dx_i}{dt},u_i)\rho J dV d\tau$, then the functional $F[x_i,u_i;t]$ can be optimize in time and varied with respect to fluid's parcel spatial coordinates, x_i, & velocity, u_i, and the product fluid density and Jacobian, ρJ, is to be optimized in time to yield the compressible fluid continuity equation and compressible Navier Stokes momentum equations, respectively.

Proof of Theorem 1. Navier Stokes Momentum equations can be derived from the Hamilton equations

Consider the following optimization of the functional, $F[x_i, u_i;\ t]$, defined by *(Eq. 6)* where the fluid's Lagrangian function, $L\ (t, x_i, \frac{dx_i}{dt}, u_i) = u_i \frac{dx_i}{dt} - \ [H]_T$, where the Hamiltonian field function, *H*, is the Hamiltonian equation defined in *(Eq. 4)* and the definition of the time field derivative operator, $\frac{d}{dt}$, on the fluid coordinates and the fluid field velocity is used as in reference [3]. For each time *t* in the interval (t_o, ∞) defines a different functional $F[x_i, u_i;\ t]$, where t is a real number not a function, therefore the functional, $F[x_i, u_i;\ t]$, needs to be optimized with respect to time and varied with respect to functions $x_i,\ and\ u_i$.

$$F[x_i, u_i; t] = \int_{t_o}^{t} \iiint_{V(t_o)} L(\tau, x_i, \frac{dx_i}{dt}, u_i)\rho J dV d\tau \tag{6}$$

The Jacobean, J, is defined as the determinant $\frac{\partial(x_1,x_2,x_3)}{\partial(Y_1,Y_2,Y_3)}$ will be optimized only with respect to time since the mass of the fluid does not change in time within the parcels' volume. The variation of the functional, F, follows from the function optimization in time (differentiation in time) and the functional optimization perform in calculus of variations [5], although these methods are equivalent of those used by Professor Morrison's seminal article [6] defining the variational derivative in page 475-476. Perform a partial differentiation to optimize functional *F* with respect to time to obtain functional *K.*

$$K[x_i, u_i; t] = \mathrm{limit}_{h\to 0} \frac{F[x_i, u_i; t+h] - F[x_i, u_i; t]}{h} = \frac{\partial F}{\partial t}$$

The functional K is given by the well-known fundamental theorem of calculus.

$$K[x_i, u_i; t] = \frac{\partial F[x_i, u_i; t]}{\partial t} = \iiint_{V(t_o)} L(t, x_i, \frac{dx_i}{dt}, u_i)\rho J dV$$

Perform the variation of the functional $K[x_i, u_i; t]$ to obtain

$$\delta K[x_i, u_i; t] = \iiint_{V(t_o)} \rho J[\delta L]_T + L[\delta_t(\rho J)]dV = 0$$

It may be deduced that $\delta \frac{\partial F[x_i,u_i;t]}{\partial t}$ might be analogous to $\frac{\delta F[x_i,u_i;\ t]}{\delta t}$ by using the variational derivative operational notation definition in Professor's Morrison's excellent article [6], although the time variation is really a partial time derivative, although this analogy must not be taken literally. Notice $[\delta L]_T = [\frac{\delta L}{\delta x_i}]_{T,u_i}\delta x_i + [\frac{\delta L}{\delta u_i}]_{x_i}\delta u_i$ since the variation of spatial coordinates and velocity occur independently, therefore the variations of the Lagrangian with respect to spatial coordinates holds the temperature and fluid velocity constant, where as the variation of the Lagrangian with respect to the fluid velocity holds the spatial coordinates constant. The optimization operator, δ_t, of the density-Jacobean product occurs with respect to time becomes a material derivative with respect to time, $\frac{d}{dt}$, times the time differential, dt.

$$\delta K[x_i, u_i; t] = \iiint_{V(t_o)} [\rho J([\frac{\delta L}{\delta x_i}]_{T,u_i}\delta x_i + [\frac{\delta L}{\delta u_i}]_{x_i}\delta u_i) + L[\frac{d(\rho J)}{dt}]dtdV = 0$$

The equation of continuity is obtain by the optimization of the product of the density, ρ, and the Jacobian[4], $J = \frac{\partial(x_1,x_2,x_3)}{\partial(Y_1,Y_2,Y_3)} \neq 0$, with respect to time only where the product $\rho J = \rho_0$ is the initial time density. Since the fluid parcel motion does not change the mass of the parcel in time, therefore the optimization is taken with respect to time, dt.

$$\frac{d(\rho J)}{dt} = J\frac{d\rho}{dt} + \rho\frac{dJ}{dt} = (J\frac{d\rho}{dt} + \rho\frac{dJ}{dt}) = \frac{d\rho_0}{dt} = 0$$

where the field or material derivative in time [3] of density and the Jacobean are given by $\frac{d\rho}{dt}$ and $\frac{dJ}{dt}$, respectively, since the advective motion of the matter needs to be taken account, so the restriction on the spatial coordinates does not apply to the material or field derivative. Since $\frac{1}{J}\frac{dJ}{dt} = \frac{\partial u_k}{\partial x_k}$ as given in reference [4] states in page, the equation results in the fluid continuity equation when there are no heat sources or sinks as seen in *(Eq. 7).*

$$\frac{d\rho}{dt} + \frac{\rho}{J}\frac{dJ}{dt} = \frac{d\rho}{dt} + \rho\frac{\partial u_k}{\partial x_k} = 0 \tag{7}$$

Therefore, the remaining variational derivative terms in the integrand below must also be zero.

$$\delta K[x_i, u_i; t] = \iiint\limits_{V(t_o)} ([\frac{\delta L}{\delta x_i}]_{T,u_i}\delta x_i + [\frac{\delta L}{\delta u_i}]_{x_i}\delta u_i)\rho J\, dV = 0$$

With added restrictions shown in brackets to make clear what's not being spatially partial differentiated, then the partial variational derivatives of the Lagrangian must be zero, $[\frac{\delta L}{\delta x_i}]_{T,u_i} = 0$, and $[\frac{\delta L}{\delta u_i}]_{x_i} = 0$, to obtain the following equations since each of the variations δx_i and δu_i are arbitrary.

$$[\frac{\delta L}{\delta x_i}]_{T,u_i} = [\frac{\partial L}{\partial x_i}]_{T,u_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \frac{dx_i}{dt}}\right) = -[\frac{\partial H}{\partial x_i}]_{T,u_i} - \frac{du_i}{dt} = 0$$

$$[\frac{\delta L}{\delta u_i}]_{x_i} = [\frac{\partial L}{\partial u_i}]_{x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \frac{du_i}{dt}}\right) = \frac{dx_i}{dt} - [\frac{\partial H}{\partial u_i}]_{x_i} - \frac{d}{dt}(0) = 0$$

Then Hamilton equation *(Eq. 8)* below defines the fluid velocity from the variation of δu_i.

$$\frac{dx_i}{dt} = [\frac{\partial H}{\partial u_i}]_{x_i} = u_i \tag{8}$$

And Hamilton equation *(Eq. 9)* from the variation of δx_i defines the compressible Navier-Stokes momentum equations again with the restriction that spatial differentiation on the potential, Ψ, occur with the temperature held fixed and the time differentiation is defined as a field derivative with respect to time [3].

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[4] The Jacobean determinant is express in terms of the current fluid parcel's center coordinates, x_i, measured from an arbitrary fixed Cartesian coordinate system, X_i, (not shown) and a fixed Cartesian coordinates, Y_i, which are dummy integration variables relative to the center of the fluid parcel's volume integral at time t_o.

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -[\frac{\partial H}{\partial x_i}]_{T,u_i} = -[\frac{\partial \Psi}{\partial x_i}]_T + \sum_{k,j=1}^{3} \frac{\partial}{\partial x_i} \int_{x_{ok}}^{x_k(t)} \frac{1}{\rho} \frac{\partial}{\partial x_j} \tau_{k,j}\, dx_k \qquad (9)$$

Since $\frac{\partial x_k(t)}{\partial x_i} = \delta_{i,k}$ results in the Kronecker delta, and since $\frac{\partial x_{ok}}{\partial x_i} = 0$, since the coordinates x_{ok} are fixed due to the initial location are being held fixed, therefore, *(Eq. 9)* becomes the compressible Navier-Stokes equations as shown in *(Eq. 11)* where the Einstein summation convention is used. The specific ideal gas law, $RT = \frac{p}{\rho}$, implies that spatial changes in potential energy per unit mass, Ψ, occurs due only to spatial changes in pressure and density while the temperature, *T*, is being held fixed, and either, the initial density, because the initial density is independent of the fixed Cartesian coordinates $\frac{\partial \rho_o}{\partial x_i} = 0$ *(Eq. 1)* since the coordinates x_{ok} are held fixed with respect to of fixed Cartesian coordinates, x_i, after some time $t > t_o$.

$$-[\frac{\partial \Psi}{\partial x_i}]_T = -\frac{\partial}{\partial x_i}(\frac{p\big(x_i(t)\big)}{\rho\big(t, x_i(t)\big)} + [RTln\left(\frac{\rho\big(t, x_i(t)\big)}{\rho(t_o, x_{oi})}\right)]_T + \phi)$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{p}{\rho^2}\frac{\partial \rho}{\partial x_i} - [\frac{RT}{\rho}\frac{\partial \rho}{\partial x_i}]_T + \frac{RT}{\rho_o}\frac{\partial \rho_o}{\partial x_i} - \frac{\partial \phi}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i}$$

Therefore, *(Eq. 10)* reads,

$$-[\frac{\partial \Psi}{\partial x_i}]_T = -\frac{\partial}{\partial x_i}(\frac{p}{\rho} + [RTln\left(\frac{\rho(t, x_i(t))}{\rho(t_o, x_{oi})}\right)]_T + \phi) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \qquad (10)$$

The negligible spatial fluctuations due to temperature are neglected for this restriction to hold true, since if the large fluctuations in temperature where present will necessitate the need to keep the temperature fluctuation terms, $-Rln\left(\frac{\rho(t,x_i(t))}{\rho(t_o,x_{oi})}\right)\frac{\partial T}{\partial x_i}$, but these temperature fluctuations *do not appear* in the compressible Navier-Stokes momentum equations. Therefore, if temperature fluctuations are negligible, then the compressible Navier–Stokes momentum equations *(Eq. 11)* can be obtain from the compressible Hamiltonian equation for viscous fluids definition via an isothermal process. For larger temperature fluctuations, then the compressible Navier-Stokes momentum equations need these temperature fluctuations to be added into the fluids momentum equations since the isothermal process does not occur, i.e. the temperature is no longer being held constant in the potential energy per unit mass, Ψ. Other more realistic gas laws, such as Van Der Waals gas equation [1] could have been used to give a more realistic approximation of the fluid in the potential energy per unit mass, Ψ, but then the ideal gas compression potential, $[RTln\left(\frac{\rho}{\rho_o}\right)]_T$, would need to be modified to match the more realistic Van Der Waals gas compression potential, if it exists.

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\tau_{k,j}\delta_{i,k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\tau_{i,j} \quad (11)$$

Since the Kronecker delta, $\delta_{i,k} = \frac{\partial x_k(t)}{\partial x_i} = 1\ if\ k = i\ or\ \delta_{i,k} =\ 0\ if\ k \neq i$, and with the understanding the spatial fluctuations in the potential energy Ψ per unit mass are only due to a isothermal processes with negligible <u>*small*</u> temperature spatial fluctuations of the fluid and with the viscous shear stress defined as $\tau_{i,j} = 2\mu e_{i,j} + \delta_{i,j}\lambda\theta$ where μ is the kinematic viscosity, $e_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the strain tensor, $\theta = \frac{\partial u_k}{\partial x_k}$ is the nonzero divergence of the fluid velocity, and λ is the second coefficient of the kinematic viscosity. Then the compressible Navier-Stokes momentum equations are given by *(Eq. 12).*

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)] + \frac{\partial}{\partial x_i}[\lambda\frac{\partial u_k}{\partial x_k}] \qquad (12)$$

QED.

If the fluid becomes incompressible and given that the kinematic viscosity, μ, is assumed to be a constant, then the incompressible Navier-Stokes momentum equations are derived from *(Eq. 12)* since the density, ρ, becomes a numeric constant, ρ_o, and because the divergence of the fluid is zero *(Eq. 13).*

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} + \frac{\mu}{\rho_o}\frac{\partial^2 u_i}{\partial x_j^2} \qquad (13)$$

3. Conclusion

This article defines a new physical model of the Bernoulli equation for compressible viscous fluid with initial time density, ρ_o, where spatial fluctuations on the scalar potential occur isothermally is based on previous research of the incompressible viscous Bernoulli equation developed in references [2,3]. This article defines a new physical model for the scalar potential energy per unit mass, $\Psi = \frac{p}{\rho} + [RTln\left(\frac{\rho}{\rho_o}\right)]_T + \phi$, of the fluid. The thesis of this article is to obtain the compressible Navier Stokes momentum equations, and fluid continuity equation from a Hamiltonian dynamical method perspective using a modified scalar potential energy per unit mass, Ψ', and a modified dissipation function. The continuity equations of the viscous fluid was also obtained by optimization with respect to time of the density-Jacobean product since the fluid parcel motion does not change the mass of the parcel in time. This article found the standard dissipation function to be *inconsistent* with the development of the Hamiltonian dynamical method; therefore the standard dissipation function, $D = \frac{1}{\rho}\tau_{k,j}\frac{\partial u_k}{\partial x_j}$, was modified to $D = \tau_{k,j}\frac{\partial}{\partial x_j}(\frac{u_k}{\rho})$ in order to obtain the compressible Navier Stokes momentum equations for viscous fluids.

If the fluid flow process is <u>*not*</u> isothermal, additional temperature fluctuation terms, $-Rln\left(\frac{\rho}{\rho_o}\right)\frac{\partial T}{\partial x_i}$, may need to be added to the viscous compressible fluid Navier Stokes momentum equations. Therefore, non-isothermal fluid experiments may be conducted to determine if these additional temperature fluctuation terms are found in the momentum equations of a fluid in which the ideal gas law is a reasonable approximation.

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