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On Zagreb Indices of Graphs with a Deleted Edge

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Abstract. In the fields of chemical graph theory, mathematical chemistry, pharmaceuticals, topological indices are calculated based on the molecular graph of a chemical compound. Topological indices are used in various fields such as in the developments of Quantitative Structure Activity Relationships (QSARs) and Quantitative Structure-Property Relationships (QSPRs). In Chemical Science, to study the physio-chemical properties of molecules most commonly used indices are Zagreb indices. In this paper, we discuss the effect of an edge deletion on Zagreb indices of simple graphs, also we verify the results obtained on some standard simple graphs such as P_n, C_n, S_n, K_n and $K_{m,n}$.

Keywords: Zagreb indices, Multiplicative Zagreb indices, Hyper Zagreb index, Multiplicative Hyper Zagreb indices, Forgotten topological index

AMS Mathematics Subject Classification (2010): 05C05, 05C07, 05C90

1. Introduction

In 1972 [1], Gutman introduced the first Zagreb index, which is among the oldest topological index. Molecular graphs are graphical models of molecules in which atoms and chemical bonds are represented by vertices and edges respectively. Many graph invariants and their applications have been studied by graph theorists and chemists. The physio-chemical properties of molecules can be studied with the help of different topological indices such as Zagreb indices. It gives an important molecular descriptor and it has been correlated with many chemical properties. Mathematical properties and applications of Zagreb multiplicative indices are discussed in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In [15], Furtula and Gutman re-introduced the forgotten topological index. The extension of studies on topological indices to fuzzy graphs can be found in [16, 17].

Let G be a connected, undirected and finite simple graph with vertex set V and edge set E. If $u \in V$ then we denote all adjacent vertices to u by $N(u)$ and cardinality of $N(u)$ by d_u, which is called the degree of vertex u. A vertex with degree one is called pendent vertex and an edge connecting pendent vertex is called a pendent edge. Graphs

$P_n, C_n, S_n, K_n, K_{m,n}$ denotes path, cycle, star, complete and complete-bipartite graphs respectively.

A chemical structure can be represented using a graph commonly known as a molecular graph or a chemical graph. In molecular graphs, atoms of a structure correspond to vertices and the bonds to edges of the graph. The topological and topographical character of the structural information of the activity characterizes structure-activity problems in the chemical field. Traditional statistical methodology requires this predictive information to be mapped to a vector space. To circumvent this vexing conversion of structural information to vector form, the edge deletion metric is required on a space of chemical graphs that defines the topology of molecules. Here we discuss the Zagreb indices and the effect of edge deletion on this graph invariant.

2. Preliminaries

In this section, we recall some definitions of Zagreb indices, which will play an important role in the subsequent proofs.

Zagreb first and second indices of simple graph G are defined as

$$M_1(G) = \sum_{uv \in E} \left[d_u + d_v \right] = \sum_{u \in V} d_u^2 \ , \ M_2(G) = \sum_{uv \in E} d_u . d_v$$

Authors [18], in 2010 introduced the first and second multiplicative Zagreb indices respectively denoted by $\Pi_1(G)$, $\Pi_2(G)$ and are defined as

$$\Pi_1(G) = \prod_{u \in V} d_u{}^2 \ , \ \Pi_2(G) = \prod_{uv \in E} d_u . d_v$$

Eliasi [19], in 2012 introduced another multiplicative-sum version of Zagreb index denoted by $\Pi_1^*(G)$ and defined as

$$\Pi_1^*(G) = \prod_{uv \in E} [du + dv]$$

In general $\Pi_1 \leq \Pi_1^*(G)$. In fact $\Pi_1 = \Pi_1^*(G)$ only if G is a cycle.

Gutman in [20] have defined the second Zagreb multiplicative index alternatively as

$$\Pi_2(G) = \prod_{u \in V} (d_u)^{d_u}$$

Recently, authors in [21] proposed new invariant Hyper Zagreb index, defined as

$$HM(G) = \sum_{uv \in E} \left(d_u + d_v \right)^2$$

Later, Kulli [22] introduced the multiplicative Hyper Zagreb indices which are defined as

$$H\Pi_1(G) = \prod_{uv \in E} \left[d_u + d_v \right]^2 \quad \text{and} \quad H\Pi_2(G) = \prod_{uv \in E} \left[d_u^2 . d_v^2 \right]$$

where d_u denotes the degree of vertex 'u'.

In [15], Furtula and Gutman re-introduced Forgotten topological index, defined as

$$F(G) = \sum_{u \in V} \left[d_u \right]^3 = \prod_{uv \in E} \left[d_u^2 + d_v^2 \right]$$

In the following section we put forward some important results on the effect of edge deletion on Zagreb indices. Also we discuss the effect in some standard graphs such as P_n, C_n, S_n, K_n and $K_{m,n}$.

3. Main Results

In this section, we study the effect on Zagreb indices, after deleting an edge from a simple graph G.

Theorem 3.1. Let $G(V,E)$ be a simple graph, $V=\{u_1,u_2,\ldots,u_n\}$. Let $e\in E$ be an edge incident to vertices u_1 and u_2 of graph G, then the effect of edge deletion on the first Zagreb index is given by

$$M_1(G-\{e\})=M_1(G)-2[d_{u_1}+d_{u_2}-1]$$

Proof: By definition, first Zagreb index is given by

$$M_1(G)=\sum_{u\in V} d_u^2=d_1^2+d_2^2+\ldots+d_{u_1}^2+d_{u_2}^2+\ldots+d_{u_n}^2$$

If $e=(u_1,u_2)$ is deleted then

$$M_1(G-\{e\})=d_1^2+d_2^2+\ldots+(d_{u_1}-1)^2+(d_{u_2}-1)^2+\ldots+d_{u_n}^2$$

$$M_1(G-\{e\})-M_1(G)=(d_{u_1}-1)^2+(d_{u_2}-1)^2-d_{u_1}^2-d_{u_2}^2$$

$$M_1(G-\{e\})=M_1(G)-2(d_{u_1}+d_{u_2}-1)$$

□

Theorem 3.2. Let $G(V,E)$ be a simple graph, with $|V|=n$ and $|E|=m$. For simplicity, as the labeling of vertices is not important, let $e\in E$ be an edge incident to vertices u_1 and u_2 of the graph G. Let $N(u_1)=\{u_2,v_1,v_2,\ldots,v_k\}$ and $N(u_2)=\{u_1,w_1,w_2,\ldots,w_l\}$, where $k+l+2\le n,$ then effect of edge deletion on the second Zagreb index is given by

$$M_2(G-\{e\})=M_2(G)-\sum_{i=1}^{k} d_{v_i}-\sum_{j=1}^{l} d_{w_j}-d_{u_1}.d_{u_2}$$

Proof: By definition, second Zagreb index is given by

$$M_2(G)=\sum_{uv\in E} d_u.d_v=d_{u_1}.d_{u_2}+\left[d_{v_1}.d_{u_1}+d_{v_2}.d_{u_1}+\ldots+d_{v_k}.d_{u_1}\right]$$
$$+\left[d_{w_1}.d_{u_2}+d_{w_2}.d_{u_2}+\ldots+d_{w_l}.d_{u_2}\right]+\mathcal{P}$$

where $\mathcal{P}$ denotes degree sum of all the non-incident edges to u_1 or u_2.

If $e=(u_1,u_2)$ is deleted then

$$M_2(G-\{e\})=\left[d_{v_1}.(d_{u_1}-1)+d_{v_2}.(d_{u_1}-1)+\ldots+d_{v_k}.(d_{u_1}-1)\right]$$
$$+\left[d_{w_1}.(d_{u_2}-1)+d_{w_2}.(d_{u_2}-1)+\ldots+d_{w_l}.(d_{u_2}-1)\right]+\mathcal{P}$$

$$M_2(G-\{e\})-M_2(G)=-d_{u_1}.d_{u_2}-[d_{v_1}-d_{v_2}-\ldots-d_{v_k}]-[d_{w_1}-d_{w_2}-\ldots-d_{w_l}]$$

$$M_2(G-\{e\})=M_2(G)-\sum_{u_1v_i\in E} d_{v_i}-\sum_{u_2w_j\in E} d_{w_j}-d_{u_1}.d_{u_2}$$

□

Theorem 3.3. Let G be a simple graph with vertex set V and edge set E, with $|V|=n$ and $|E|=m$. Let $e\in E$ be an edge incident to vertices u_1 and u_2 of the graph G, then the effect of edge deletion on the first multiplicative Zagreb index is given by

$$\frac{\Pi_1\left(G-\{e\}\right)}{\Pi_1(G)}=\frac{[d_{u_1}-1]^2.\,[d_{u_2}-1]^2}{d_{u_1}^2.d_{u_2}^2}$$

Proof: We have

$$\Pi_1(G)=\prod_{u\in V} d_u{}^2=d_{u_1}^2.d_{u_2}^2.d_{u_3}^2\ldots d_{u_n}^2.$$

If an edge $'e'$ joining vertices u_1 and u_2 is deleted then their degrees decreases by 1.

$$\Pi_1\left(G-\{e\}\right)=(d_{u_1}-1)^2.(d_{u_2}-1)^2.d_{u_3}^2\ldots d_{u_n}^2$$

hence
$$\frac{\Pi_1\left(G-\{e\}\right)}{\Pi_1(G)}=\frac{[d_{u_1}-1]^2.\,[d_{u_2}-1]^2}{d_{u_1}^2.d_{u_2}^2}$$

□

Theorem 3.4. Let $G(V,E)$ be a simple graph. Let $e\in E$ be an edge incident to vertices u_1 and u_2 of the graph $G(V,E)$. Let $N(u_1)$ = { u_2, v_1, v_2, … , v_k } and $N(u_2)$ ={ u_1, w_1, w_2, … , w_l } then the effect of edge deletion on the second multiplicative Zagreb index is given by

$$\frac{\Pi_2\left(G-\{e\}\right)}{\Pi_2(G)}=\frac{(d_{u_1}-1)^{(d_{u_1}-1)}.\,(d_{u_2}-1)^{(d_{u_2}-1)}}{d_{u_1}^{d_{u_1}}.\,d_{u_2}^{d_{u_2}}}$$

Proof: We have

$$\Pi_2(G)=\prod_{u\in V}(d_u)^{d_u}=d_{u_1}^{d_{u1}}.d_{u_2}^{d_{u2}}.d_{u_3}^{d_{u3}}\ldots d_{u_n}^{d_{un}}$$

If an edge adjoining vertices u_1 and u_2 is deleted then their degrees decreases by 1.

$$\Pi_2\left(G-\{e\}\right)=(d_{u_1}-1)^{(d_{u_1}-1)}.(d_{u_2}-1)^{(d_{u_2}-1)}.d_{u_3}^{d_{u_3}}\ldots d_{u_n}^{d_{u_n}}$$

hence
$$\frac{\Pi_2\left(G-\{e\}\right)}{\Pi_2(G)}=\frac{(d_{u_1}-1)^{(d_{u_1}-1)}.\,(d_{u_2}-1)^{(d_{u_2}-1)}}{d_{u_1}^{d_{u_1}}.\,d_{u_2}^{d_{u_2}}}$$

□

Theorem 3.5. Let $G(V,E)$ be a simple graph. Let $e\in E$ be an edge incident to vertices u_1 and u_2 of the graph $G(V,E)$. Let $N(u_1)$ = { u_2, v_1, v_2, … , v_k } and $N(u_2)$ ={ u_1, w_1, w_2, … , w_l }, then the effect of edge deletion on the multiplicative-sum Zagreb index is given by

$$\frac{\Pi_1^*\left(G-\{e\}\right)}{\Pi_1^*(G)}=\prod_{i=1}^{k}\left[1-\frac{1}{d_{u_1}+d_{v_i}}\right].\prod_{j=1}^{l}\left[1-\frac{1}{d_{u_2}+d_{w_j}}\right].\left[\frac{1}{d_{u_1}+d_{u_2}}\right]$$

Proof: By definition

$$\Pi_1^*(G)=\prod_{uv\in E}(d_u+d_v)$$

$$\Pi_1^*\left(G-\{e\}\right)=\prod_{i=1}^{k}(d_{u_1}+d_{v_i}-1).\prod_{j=1}^{l}(d_{u_2}+d_{v_j}-1)$$

$$\frac{\Pi_1^*\left(G-\{e\}\right)}{\Pi_1^*\left(G\right)}=\frac{U_1.U_2.\mathcal{P}}{U_1^{'}.U_2^{'}.[d_{u_1}+d_{u_2}].\mathcal{P}}$$

where

$$U_1=(d_{u_1}+d_{v_1}-1)(d_{u_1}+d_{v_2}-1)\ldots(d_{u_1}+d_{v_k}-1)$$

$$U_2=(d_{u_2}+d_{w_1}-1)(d_{u_2}+d_{w_2}-1)\ldots(d_{u_2}+d_{w_l}-1)$$

$$U_1^{'}=(d_{u_1}+d_{v_1})(d_{u_1}+d_{v_2})\ldots(d_{u_1}+d_{v_k})$$

$$U_2^{'}=(d_{u_2}+d_{w_1})(d_{u_2}+d_{w_2})\ldots(d_{u_2}+d_{w_l})$$

$\mathcal{P}$ denotes multiplicative degree sum of all the non-incident edges to u_1 or u_2.

$$\frac{\Pi_1^*\left(G-\{e\}\right)}{\Pi_1^*(G)}=\left(1-\frac{1}{d_{u_1}+d_{v_1}}\right)\left(1-\frac{1}{d_{u_1}+d_{v_2}}\right)\ldots\left(1-\frac{1}{d_{u_1}+d_{v_k}}\right)$$

$$.\left(1-\frac{1}{d_{u_2}+d_{w_1}}\right)\left(1-\frac{1}{d_{u_2}+d_{w_2}}\right)\ldots\left(1-\frac{1}{d_{u_2}+d_{w_l}}\right).\left(1-\frac{1}{d_{u_1}+d_{u_2}}\right)$$

$$=\prod_{i=1}^{k}\left(1-\frac{1}{d_{u_1}+d_{v_i}}\right).\prod_{j=1}^{l}\left(1-\frac{1}{d_{u_2}+d_{w_j}}\right).\left(\frac{1}{d_{u_1}+d_{u_2}}\right)$$

□

Theorem 3.6. Let $G(V,E)$ be a simple graph. $V=\{u_1,u_2,\ldots,u_n\}$. Let $e\in E$ be an edge incident to vertices u_1 and u_2 of the graph G, then the effect of edge deletion on the forgotten topological index is given by

$$F(G-\{e\})=F(G)+3[d_{u_1}^2+d_{u_2}^2-d_{u_1}-d_{u_2}]+2$$

Proof: By definition, first Zagreb index is given by

$$F(G)=\sum_{u\in V}d_u^3=d_{u_1}^3+d_{u_2}^3+d_{u_3}^3+\ldots+d_{u_n}^3$$

If $e=(u_1,u_2)$ is deleted then

$$F(G-\{e\})=(d_{u_1}-1)^3+(d_{u_2}-1)^3+d_{u_3}^3+\ldots+d_{u_n}^3$$

$$F(G-\{e\})-F(G)=(d_{u_1}-1)^3+(d_{u_2}-1)^3-d_{u_1}^3-d_{u_2}^3$$

$$F(G-\{e\}) = F(G) - 3(d_{u_1}^2 + d_{u_2}^2 - d_{u_1} - d_{u_2}) - 2$$

□

Theorem 3.7. Let G be a simple graph with vertex set V and edge set E, with $|V|=n$ and $|E|=m$. Let $e\in E$ be an edge incident to vertices u_1 and u_2 of graph G. Let $N(u_1) = \{u_2, v_1, v_2, \dots, v_k\}$ and $N(u_2) = \{u_1, w_1, w_2, \dots, w_l\}$, where $k+l+2\le n$, then the effect of edge deletion on the Hyper Zagreb index is given by

$$HM(G-\{e\}) = HM(G) + 3[d_{u_1} + d_{u_2}] - 2[d_{u_1}^2 + d_{u_2}^2] - [d_{u_1} + d_{u_2}]^2 - 2\sum_{i=1}^{k} d_{v_i} - 2\sum_{j=1}^{l} d_{w_j} - 2$$

Proof: By definition, hyper Zagreb index is given as

$$HM(G) = \sum_{uv\in E} (d_u + d_v)^2$$

$$HM(G-\{e\}) = [(d_{u_1} + d_{v_1} - 1)^2 + (d_{u_1} + d_{v_2} - 1)^2 + \dots + (d_{u_1} + d_{v_k} - 1)^2] + [(d_{u_2} + d_{w_1} - 1)^2 + (d_{u_2} + d_{w_2} - 1)^2 + \dots + (d_{u_2} + d_{w_l} - 1)^2] + \mathcal{P}$$

where $\mathcal{P}$ denotes sum of square of degree sum of all the non-incident edges to u_1 or u_2. Hence

$$HM(G-\{e\}) - HM(G) = \sum_{i=1}^{k}\left[d_{v_i} + d_{u_1} - 1\right]^2 + \sum_{j=1}^{l}\left[d_{w_j} + d_{u_2} - 1\right]^2 - \left[d_{u_1} + d_{u_2}\right]^2 - \sum_{i=1}^{k}\left[d_{v_i} + d_{u_1}\right]^2 + \sum_{j=1}^{l}\left[d_{w_j} + d_{u_2}\right]^2$$

$$HM(G-\{e\}) = HM(G) + 3[d_{u_1} + d_{u_2}] - 2[d_{u_1}^2 + d_{u_2}^2] - [d_{u_1} + d_{u_2}]^2 - 2\sum_{i=1}^{k} d_{v_i} - 2\sum_{j=1}^{l} d_{w_j} - 2$$

□

Theorem 3.8. Let $G(V,E)$ be a simple graph. Let $e\in E$ be an edge incident to vertices u_1 and u_2 of graph $G(V,E)$. Let $N(u_1) = \{u_2, v_1, v_2, \dots, v_k\}$ and $N(u_2) = \{u_1, w_1, w_2, \dots, w_l\}$, then the effect of edge deletion on the first multiplicative hyper Zagreb index is given by

$$\frac{H\Pi_1(G-\{e\})}{H\Pi_1(G)} = \prod_{i=1}^{k}\left(1 - \frac{1}{d_{u_1} + d_{v_i}}\right)^2 . \prod_{j=1}^{l}\left(1 - \frac{1}{d_{u_2} + d_{w_j}}\right)^2 . \left(\frac{1}{d_{u_1} + d_{u_2}}\right)^2$$

Proof: By definition first multiplicative hyper Zagreb index is defined as

$$H\Pi_1(G) = \prod_{uv\in E} (d_u + d_v)^2$$

$$H\Pi_1\left(G-\{e\}\right)=\prod_{i=1}^{k}(d_{u_1}+d_{v_i}-1)^2.\prod_{j=1}^{l}(d_{u_2}+d_{v_j}-1)^2$$

$$\frac{H\Pi_1\left(G-\{e\}\right)}{H\Pi_1\left(G\right)}=\frac{U_1.U_2.\mathcal{P}}{U_1^{'}.U_2^{'}.[d_{u_1}+d_{u_2}]^2.\mathcal{P}}$$

where

$$U_1=(d_{u_1}+d_{v_1}-1)^2\,(d_{u_1}+d_{v_2}-1)^2\ldots(d_{u_1}+d_{v_k}-1)^2$$

$$U_2=(d_{u_2}+d_{w_1}-1)^2\,(d_{u_2}+d_{w_2}-1)^2\ldots(d_{u_2}+d_{w_l}-1)^2$$

$$U_1^{'}=(d_{u_1}+d_{v_1})^2\,(d_{u_1}+d_{v_2})^2\ldots(d_{u_1}+d_{v_k})^2$$

$$U_2^{'}=(d_{u_2}+d_{w_1})^2\,(d_{u_2}+d_{w_2})^2\ldots(d_{u_2}+d_{w_l})^2$$

$\mathcal{P}$ denotes multiplicative degree sum of all the non-incident edges to u_1 or u_2.

$$\frac{H\Pi_1\left(G-\{e\}\right)}{H\Pi_1\left(G\right)}=\left(1-\frac{1}{d_{u_1}+d_{v_1}}\right)^2\left(1-\frac{1}{d_{u_1}+d_{v_2}}\right)^2\ldots\left(1-\frac{1}{d_{u_1}+d_{v_k}}\right)^2$$

$$.\left(1-\frac{1}{d_{u_2}+d_{w_1}}\right)^2\left(1-\frac{1}{d_{u_2}+d_{w_2}}\right)^2\ldots\left(1-\frac{1}{d_{u_2}+d_{w_l}}\right)^2.$$

$$.\left(1-\frac{1}{d_{u_1}+d_{u_2}}\right)^2$$

$$=\prod_{i=1}^{k}\left(1-\frac{1}{d_{u_1}+d_{v_i}}\right)^2.\prod_{j=1}^{l}\left(1-\frac{1}{d_{u_2}+d_{w_j}}\right)^2.\left(\frac{1}{d_{u_1}+d_{u_2}}\right)^2$$

□

Theorem 3.9. Let $G(V,E)$ be a simple graph. Let $e\in E$ be an edge incident to vertices u_1 and u_2 of the graph $G(V,E)$. Let $N(u_1)$ = { $u_2, v_1, v_2, \ldots, v_k$ } and $N(u_2)$ ={ $u_1, w_1, w_2, \ldots, w_l$ }, then the effect of edge deletion on the second multiplicative hyper Zagreb index is given by

$$\frac{H\Pi_2\left(G-\{e\}\right)}{H\Pi_2(G)}=\frac{(d_{u_1}-1)^{2(d_{u_1}-1)}.(d_{u_2}-1)^{2(d_{u_2}-1)}}{d_{u_1}^{2d_{u_1}}.d_{u_2}^{2d_{u_2}}}$$

Proof: By definition, second multiplicative hyper Zagreb index is defined as

$$H\Pi_2(G)=\prod_{uv\in E}(d_u.d_v)^2$$

$$H\Pi_2(G-\{e\})=\prod_{i=1}^{k}\left[d_{v_i}^2.(d_{u_1}-1)^2\right].\prod_{j=1}^{l}\left[d_{w_j}^2.(d_{u_2}-1)^2\right].\mathcal{P}$$

where $\mathcal{P}$ denotes multiplicative degree sum of all the non-incident edges to u_1 or u_2.

$$\frac{H\Pi_2\left(G-\{e\}\right)}{H\Pi_2\left(G\right)}=\frac{U_1.U_2.\mathcal{P}}{U_1^{'}.U_2^{'}.[d_{u_1}.d_{u_2}]^2.\mathcal{P}}$$

where,

$$U_1=d_{v_1}^2(d_{u_1}-1)^2.d_{v_2}^2(d_{u_1}-1)^2\ldots d_{v_k}^2(d_{u_1}-1)^2$$
$$U_2=d_{w_1}^2(d_{u_2}-1)^2.d_{w_2}^2(d_{u_2}-1)^2\ldots d_{w_l}^2(d_{u_2}-1)^2$$
$$U_1^{'}=(d_{u_1}.d_{v_1})^2.(d_{u_1}.d_{v_2})^2\ldots(d_{u_1}.d_{v_k})^2$$
$$U_2^{'}=(d_{u_2}.d_{w_1})^2.(d_{u_2}.d_{w_2})^2\ldots(d_{u_2}.d_{w_l})^2$$

$\mathcal{P}$ denotes multiplicative degree sum of all the non-incident edges to u_1 or u_2.

$$\frac{H\Pi_2\left(G-\{e\}\right)}{H\Pi_2\left(G\right)}=\frac{(d_{u_1}-1)^{2k}.(d_{u_2}-1)^{2l}}{(d_{u_1})^{2k}.(d_{u_2})^{2l}.[d_{u_1}.d_{u_2}]^2}$$
$$=\frac{(d_{u_1}-1)^{2(d_{u_1}-1)}.(d_{u_2}-1)^{2(d_{u_2}-1)}}{d_{u_1}^{2d_{u_1}}.d_{u_2}^{2d_{u_2}}}$$

□

4. Deleting an edge from some standard graphs

Here we discuss the effect on Zagreb indices after deletion an edge from some standard graphs such as P_n, C_n, S_n, K_n and $K_{m,n}$ and verify the effect using the results obtained. Let $e=(i,j)$ denotes an edge joining end vertices with degrees $'i'$ and $'j'$.

4.1. Path graph P_n

In case of path graph $'P_n'$; $n\geq 4$, there are two types of edges $e_1(2,1)$ and $e_2(2,2)$

i) $M_1(P_n)=4n-6$

$M_1(P_n-\{e_1\})=4n-10=(4n-6)-2(2+1-1)$

$M_1(P_n-\{e_2\})=4n-12=(4n-6)-2(2+2-1)$

ii) $M_2(P_n)=4n-8$

$M_2(P_n-\{e_1\})=4n-12=(4n-8)-2-0-2$

$M_2(P_n-\{e_2\})=4n-16=(4n-8)-2-2-4$

iii) $\Pi_1(P_n)=(2)^{2(n-2)}$

$\Pi_1(P_n-\{e_1\})=0$

$\Pi_1(P_n-\{e_2\})=(2)^{2(n-4)}=\frac{1}{2^2.2^2}.(2)^{2(n-2)}$

iv) $\Pi_2(P_n)=(2)^{2(n-2)}$

$\Pi_2(P_n-\{e_1\})=0$

$\Pi_2(P_n-\{e_2\})=(2)^{2(n-4)}=\frac{1}{2^2.2^2}.(2)^{2(n-2)}$

v) $\Pi_1^*(P_n)=3^2.(2)^{2(n-3)}$

$\Pi_1^*(P_n-\{e_1\})=3^2.(2)^{2(n-4)}=\frac{3}{4}.\frac{3^2}{3}.(2)^{2(n-3)}$

$\Pi_1^*(P_n-\{e_2\})=3^4.(2)^{2(n-6)}=\frac{3}{4}.\frac{3}{4}.\frac{3^2}{4}.(2)^{2(n-3)}$

vi) $F(P_n)=8n-22$

$F(P_n-\{e_1\})=8n-30=(8n-22)-3(2^2+1-2-1)-2$

$F(P_n-\{e_2\})=8n-36=(8n-22)-3(2^2+2^2-2-2)-2$

vii) $HM(P_n)=16n-30$

$HM(P_n-\{e_1\})=16n-46 \;=(16n-30)+3(2+1)-2(4+1)-6-(2+1)^2$

$HM(P_n-\{e_2\})=16n-60 \;=(16n-30)+3(4)-2(8)-(4)^2-8-2$

viii) $H\Pi_1(P_n)=3^4.(4)^{2(n-3)}$

$H\Pi_1(P_n-\{e_1\})=3^4.(4)^{2(n-4)}=3^4.(4)^{2(n-3)}.\frac{1}{9}.\frac{9}{16}$

$H\Pi_1(P_n-\{e_2\})=3^8.(4)^{2(n-6)}=3^4.(4)^{2(n-3)}.\frac{1}{16}.\frac{9}{16}.\frac{9}{16}$

ix) $H\Pi_2(P_n)=(4)^{2(n-2)}$

$H\Pi_2(P_n-\{e_1\})=0$

$H\Pi_2(P_n-\{e_2\})=(4)^{2(n-4)}=(4)^{2(n-2)}.\frac{1}{2^4.2^4}$

4.2. Cycle graph C_n

In case of cycle graph 'C_n', any edge is of type $e(2,2)$

i) $M_1(C_n)=4n$

$M_1(C_n-\{e\})=4n-6=4n-2(2+2-1)$

ii) $M_2(C_n)=4n$

$M_2(C_n-\{e\})=4n-8=4n-2-2-4$

iii) $\Pi_1(C_n)=(2)^{2n}$

$\Pi_1(C_n-\{e\})=(2)^{2(n-2)}=2^{2n}.\frac{1}{2^2.2^2}$

iv) $\Pi_2(C_n)=(2)^{2n}$

$\Pi_2(C_n - \{e\}) = (2)^{2(n-2)} = 2^{2n}.\frac{1}{2^2.2^2}$

v) $\Pi_1^*(C_n) = (2)^{2n}$

$\Pi_1^*(C_n - \{e\}) = 3^2.(2)^{2(n-3)} = 2^{2n}.\frac{3}{4}.\frac{3}{4}.\frac{1}{4}$

vi) $F(C_n) = 8n$

$F(C_n - \{e\}) = 8n - 14 = 8n - 3(4+4-2-2) - 2$

vii) $HM(C_n) = 16n$

$HM(C_n - \{e\}) = 16n - 30 = 16n + 3(2+2) - 2(8) - 4^2 - 4 - 4 - 2$

viii) $H\Pi_1(C_n) = (4)^{2n}$

$H\Pi_1(C_n - \{e\}) = 3^4.(4)^{2(n-3)} = 4^{2n}.\frac{9}{16}.\frac{9}{16}.\frac{1}{16}$

ix) $H\Pi_2(C_n) = (4)^{2n}$

$H\Pi_2(C_n - \{e\}) = (4)^{2(n-2)} = 4^{2n}.\frac{1}{16}.\frac{1}{16}$

4.3. Star graph S_n

In case of star graph $'S_n'$ with $n(\geq 4)$, any edge is of type $e(n-1, 1)$

i) $M_1(S_n) = n(n-1)$

$M_1(S_n - \{e\}) = (n-1)(n-2) = n(n-1) - 2(n-1)$

ii) $M_2(S_n) = (n-1)^2$

$M_2(S_n - \{e\}) = (n-2)^2 = (n-1)^2 - (n-2) - (n-1)$

iii) $\Pi_1(S_n) = (n-1)^2$

$\Pi_1(S_n - \{e\}) = 0 = (n-1)^2.0$

iv) $\Pi_2(S_n) = (n-1)^{(n-1)}$

$\Pi_2(S_n - \{e\}) = 0 = (n-1)^{(n-1)}.0$

v) $\Pi_1^*(S_n) = (n)^{(n-1)}$

$\Pi_1^*(S_n - \{e\}) = (n-1)^{(n-2)} = n^{(n-1)}.\left(1 - \frac{1}{n}\right)^{(n-2)}.\left(\frac{1}{n}\right)$

vi) $F(S_n) = (n-1) + (n-1)^3$

$F(S_n - \{e\}) = (n-2) + (n-2)^3$

$= (n-1) + (n-1)^3 - 3\left[(n-1)^2 - (n-1)\right] - 2$

vii) $HM(S_n) = n^2(n-1)$

$$HM(S_n - \{e\}) = (n-1)^2(n-2)$$
$$= n^2(n-1) + 3n - 2\left[(n-1)^2 + 1\right] - n^2 - 2(n-2) - 2$$

viii) $H\Pi_1(S_n) = n^{2(n-1)}$

$$H\Pi_1(S_n - \{e\}) = (n-1)^{2(n-2)} = n^{2(n-1)}.\left(1 - \frac{1}{n}\right)^{2(n-2)}.\frac{1}{n^2}$$

ix) $H\Pi_2(S_n) = (n-1)^{2(n-1)}$

$$H\Pi_2(S_n - \{e\}) = (n-2)^{2(n-2)} = (n-1)^{2(n-1)}.\frac{(n-2)^{2(n-2)}}{(n-1)^{2(n-1)}}$$

4.4. Complete graph K_n

In case of complete graph 'K_n', any edge is of type $e(n-1, n-1)$

i) $M_1(K_n) = n(n-1)^2$

$$M_1(K_n - \{e\}) = n^3 - 2n^2 - 3n + 6 = n(n-1)^2 - 2(2n-3)$$

ii) $M_2(K_n) = \dfrac{n(n-1)^3}{2}$

$$M_2(K_n - \{e\}) = \frac{(n-1)(n-2)(n^2-5)}{2} = \frac{n(n-1)^3}{2} - 2(n-1)(n-2) - (n-1)^2$$

iii) $\Pi_1(K_n) = (n-1)^{2n}$

$$\Pi_1(K_n - \{e\}) = (n-1)^{2(n-2)}.(n-2)^4 = (n-1)^{2n}.\frac{(n-2)^4}{(n-1)^4}$$

iv) $\Pi_2(K_n) = (n-1)^{n(n-1)}$

$$\Pi_2(K_n - \{e\}) = (n-1)^{(n-1)(n-2)}.(n-2)^{2(n-2)} = (n-1)^{n(n-1)}.\frac{(n-2)^{2(n-2)}}{(n-1)^{2(n-1)}}$$

v) $\Pi_1^* = (2n-2)^{n(n-1)/2}$

$$\Pi_1^*(K_n - \{e\}) = (2n-2)^{(n-2)(n-3)/2}.(2n-3)^{2(n-2)}$$
$$= (2n-2)^{n(n-1)/2}.\left(\frac{2n-3}{2n-2}\right)^{2(n-2)}.\left(\frac{1}{2n-2}\right)$$

vi) $F(K_n) = n(n-1)^3$

$$F(K_n - \{e\}) = (n-1)^3(n-2) + 2(n-2)^3 = n(n-1)^3 - 6[(n-1)^2 - (n-1)] - 2$$

vii) $HM(K_n) = 2n(n-1)^3$

$$HM(K_n - \{e\}) = 2(n^4 - 3n^3 - 3n^2 + 14n - 10)$$
$$= 2n(n-1)^3 + 6(n-1) - 12(n-1)^2 - 2$$

viii) $H\Pi_1(K_n) = (2n-2)^{n(n-1)}$

$H\Pi_1(K_n - \{e\}) = (2n-3)^{4(n-2)}.(2n-2)^{(n^2-5n+6)}$

$$= (2n-2)^{n(n-1)}.\left(\frac{2n-3}{2n-2}\right)^{4(n-2)}.\left(\frac{1}{2n-2}\right)^2$$

ix) $H\Pi_2(K_n) = (n-1)^{2n(n-1)}$

$$H\Pi_2(K_n - \{e\}) = (n-1)^{2(n-1)(n-2)}.(n-2)^{4(n-2)} = (n-1)^{2n(n-1)}.\frac{(n-2)^{4(n-2)}}{(n-1)^{4(n-1)}}$$

4.5. Complete bipartite graph $K_{m,n}$

In case of complete bipartite graph '$K_{m,n}$' ; with $m,n \geq 2$, any edge is of type $e(n,m)$

i) $M_1(K_{m,n}) = (mn).(m+n)$

$M_1(K_{m,n} - \{e\}) = (m+n).(mn-2) + 2 = mn.(m+n) - 2(m+n-1)$

ii) $M_2(K_{m,n}) = m^2.n^2$

$M_2(K_{m,n} - \{e\}) = m^2n^2 - 3mn + m + n = m^2n^2 - m(n-1) - n(m-1) - mn$

iii) $\Pi_1(K_{m,n}) = n^{2m}.m^{2n}$

$\Pi_1(K_{m,n} - \{e\}) = (m-1)^2.(n-1)^2.m^{2(n-1)}.n^{2(m-1)}$

$$= n^{2m}.m^{2n}.\frac{(n-1)^2.(m-1)^2}{m^2.n^2}$$

iv) $\Pi_2(K_{m,n}) = n^{m\,n}.m^{m\,n}$

$\Pi_2(K_{m,n} - \{e\}) = (m-1)^{(m-1)}.(n-1)^{(n-1)}.m^{m\,(n-1)}.n^{n\,(m-1)}$

$$= n^{m\,n}.m^{n\,m}.\frac{(m-1)^{(m-1)}.(n-1)^{(n-1)}}{m^m.n^n}$$

v) $\Pi_1^*(K_{m,n}) = (m+n)^{(m\,n)}$

$\Pi_1^*(K_{m,n} - \{e\}) = (m+n-1)^{(m+n-2)}.(m+n)^{(m\,n-m-n+1)}$

$$= (m+n)^{m\,n}.\frac{(m+n-1)^{(n-1)}.(m+n-1)^{(m-1)}}{(m+n)^{(m+n-1)}}$$

vi) $F(K_{m,n}) = m.n^3 + n.m^3$

$F(K_{m,n} - \{e\}) = (m-1)^3 + (n-1)^3 + m^3(n-1) + n^3(m-1)$

$$= m.n^3 + n.m^3 - 3(m^2 + n^2 - m - n) - 2$$

vii) $HM(K_{m,n}) = (mn)(m+n)^2$

$$HM(K_{m,n}-\{e\})=(m+n-1)^2(m+n-2)+(m+n)^2(mn-n-m+1)$$
$$=mn.(m+n)^2+3(m+n)-2(m^2+n^2)$$
$$-(m+n)^2-2m(n-1)-2n(m-1)-2$$

viii) $H\Pi_1(K_{m,n})=(m+n)^{2(m\,n)}$

$$H\Pi_1(K_{m,n}-\{e\})=(m+n-1)^{2(m+n-2)}.(m+n)^{2(mn-n-m+1)}$$
$$=(m+n)^{2m\,n}.\left(\frac{m+n-1}{m+n}\right)^{2(n-1)}$$
$$.\left(\frac{m+n-1}{m+n}\right)^{2(m-1)}.\frac{1}{(m+n)^2}$$

ix) $H\Pi_2(K_{m,n})=(mn)^{2m\,n}$

$$H\Pi_2(K_{m,n}-\{e\})=[(m-1)n]^{2(m-1)}.[m(n-1)]^{2(n-1)}(m^2n^2).(mn-n-m+1)$$
$$=(mn)^{2m\,n}.\frac{(n-1)^{2(n-1)}.(m-1)^{2(m-1)}}{n^{2n}.m^{2m}}$$

5. Conclusion

We have studied the properties of Zagreb indices and the effect on them when an edge is deleted from a given graph. We have also verified the results obtained with some of the standard graphs by taking into consideration all the possible cases. More properties and applications of Zagreb indices will be discussed in the following papers.

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